Describing Polynomials as Equivalent to Explicit Solutions

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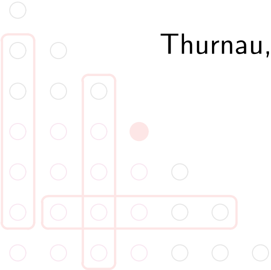

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Observation

Polynomials $P(X_1) \neq 0$ of degree $\deg(P) < d_1$ have

fewer than d_1 zeros.

Polynomials $P(X_1)$ of degree $\deg(P) < d_1$ have

– on $d_1 + 1$ given points – not exactly one nonzero.

Polynomials $P(X_1, \ldots, X_n)$ of total degree $\deg(P) < d_1 + \cdots + d_n$ have

– on grids $\mathfrak{X} = \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n$ of $(d_1 + 1) \times \cdots \times (d_n + 1)$ points –

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Definition

Assume $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ and let $\mathcal{R}$ be an integral domain.

A *d-grid* $\mathfrak{X} \subseteq \mathcal{R}^n$ is a *grid* $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n$ of size $(d_1 + 1) \times \cdots \times (d_n + 1)$.

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Corollary ($\neq$1 -Theorem)

If $\deg(P) < \Sigma d$ *, then*

$$\Big|\{\, x \in \mathfrak{X} \;\big|\; P(x) \neq 0 \,\}\Big| \neq 1 .$$

Corollary (Generalized Chevalley-Warning-Theorem)

Let $\mathcal{R} := \mathbb{F}_q$ *and* $P_1, \ldots, P_m \in \mathcal{R}[X_1, \ldots, X_n]$.

If $(q-1)\sum_i \deg(P_i) < \Sigma d$ *, then*

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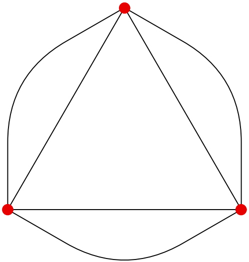

A 4-regular graph without 3-regular subgraph

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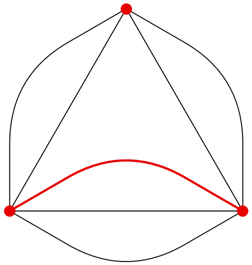

Extended graph

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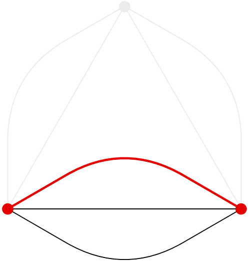

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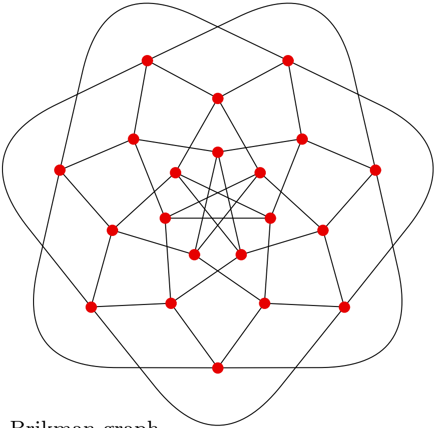

Brikman graph

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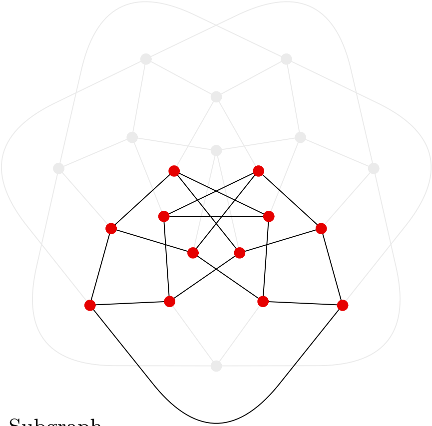

Subgraph

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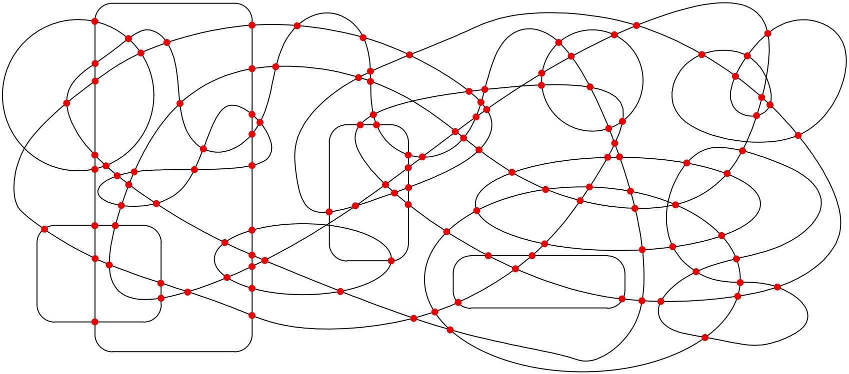

An other 4-regular graph

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I am sure there is one!

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Every $\overbrace{\textit{loopless 4-regular multigraph plus one edge}}^{\textit{extended 4-regular graph}}$ $G = (V, \overbrace{E \uplus \{e_0\}}^{\bar{E}})$ *contains a nontrivial 3-regular subgraph.*

Proof.

The subgraphs $\mathcal{S} \subseteq \bar{E}$ correspond to the points $x = (x_e)_{e \in \bar{E}}$ of the Boolean grid $\mathfrak{X} := \{0,1\}^{\bar{E}} \subseteq \mathbb{F}_3^{\bar{E}}$.

$$\left(\begin{array}{l} \bar{E} \longleftrightarrow \{1,\ldots,n\} \\ V \longleftrightarrow \{1,\ldots,m\} \end{array} \right)$$

Algebraic Solution:

$$P_v := \sum_{e \ni v} X_e \in \mathbb{F}_3[\, X_e \mid e \in \bar{E} \,] \quad \text{for all } v \in V .$$

Degree Restriction: $(3-1)\sum_v \deg(P_v) = 2|V| = |E| < |\bar{E}| = \Sigma d$.

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Advantages / Disadvantages of Algebraic Solutions

1. Indirect proof, we do not obtain explicit solutions.
   (Just exponential time algorithms.)
2. Sometimes, easy to find.
3. Sometimes, infinitely many algebraic solutions fit into a general form and can be presented in just one line.
   ($\longrightarrow$ "Finite Blackboard Problem")
4. Only the primes occur as characteristic of finite fields.
   ($\longrightarrow$ Olson's Theorem; Alon, Friedland, Kalai's Conjecture. (Appendix))

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Theorem (Coefficient Formula)

Let $\mathfrak{X} \subseteq \mathcal{R}^n$ *be a* *d-grid*.

For polynomials $P = \sum_{\delta \in \mathbb{N}^n} P_\delta X^\delta \in \mathcal{R}[X_1, \ldots, X_n]$ *of total degree* $\deg(P) \leq \Sigma d := \sum_j d_j$,

$$\boxed{P_d = \Sigma(N^{-1} P|_{\mathfrak{X}})} \quad := \sum_{x \in \mathfrak{X}} N(x)^{-1} P(x) \ ,$$

where the maps $N, P|_{\mathfrak{X}} : \mathfrak{X} \longrightarrow \mathcal{R}$ *are defined by*
$N(x_1, \ldots, x_n) := \prod_j \prod_{\xi \in \mathfrak{X}_j \setminus \{x_j\}} (x_j - \xi)$ *and* $P|_{\mathfrak{X}}(x) := P(x)$.

Corollary (Combinatorial Nullstellensatz (Alon, Tarsi))

If $\deg(P) \leq \Sigma d$ *, then*

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$N(x_1, \ldots, x_n) := \prod_j \prod_{\xi \in \mathfrak{X}_j \setminus \{x_j\}} (x_j - \xi)$ *and* $P|_{\mathfrak{X}}(x) := P(x)$.

Corollary (Combinatorial Nullstellensatz (Alon, Tarsi))

If $\deg(P) \leq \Sigma d$, *then*

$$\boxed{P_d \neq 0 \implies P|_{\mathfrak{X}} \neq 0} \ .$$

Corollary ($\neq$1 -Theorem)

If $\deg(P) < \Sigma d$, *then*

$$\boxed{\left|\{\, x \in \mathfrak{X} \;\vdots\; P(x) \neq 0 \,\}\right| \;\neq\; 1} \,.$$

Corollary (Generalized Chevalley-Warning-Theorem)

Let $\mathcal{R} := \mathbb{F}_q$ *and* $P_1, \ldots, P_m \in \mathcal{R}[X_1, \ldots, X_n]$.

If $(q-1)\sum_i \deg(P_i) < \Sigma d$, *then*

$$\boxed{\left|\{\, x \in \mathfrak{X} \;\vdots\; P_1(x) = \cdots = P_m(x) = 0 \,\}\right| \;\neq\; 1} \,.$$

Proof.

Define $P := \prod_{i=1}^{m}(1 - P_i^{\,q-1})$ and apply the $\neq$1 -Theorem. □

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Theorem (Interpolation Formula)

Let $\mathfrak{X} \subseteq \mathbb{F}^n$ be a d-grid over a field $\mathbb{F}$ and $y\colon \mathfrak{X} \longrightarrow \mathbb{F}$ a map.

There exists a unique polynomial $P \in \mathbb{F}[X_1, \ldots, X_n]$ with partial degrees $\deg_j(P) \le d_j$ that interpolates y, i.e., $P|_{\mathfrak{X}} = y$.

The coefficients P_δ of P are given by

$$\boxed{P_\delta = \Sigma(M_\delta\, y)}\ ,$$

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Polynomials $P \in \mathcal{R}[X_1, \ldots, X_n]$ with partial degrees $\deg_j(P) \le d_j$ are uniquely determined by $P|_{\mathfrak{X}}$. The coefficients P_δ of P are given by

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Proof of the Coefficient Formula.

Transform P into a *trimmed* polynomial $P/\mathfrak{X}$ with

(i) $(P/\mathfrak{X})|_{\mathfrak{x}} = P|_{\mathfrak{x}}$,

(ii) $(P/\mathfrak{X})_d = P_d$,

(iii) $\deg_j(P/\mathfrak{X}) \leq d_j$ for $j = 1, \ldots, n$.

Then

$$P_d \overset{(ii)}{=} (P/\mathfrak{X})_d \overset{(iii)}{=} \Sigma(M_d\,(P/\mathfrak{X})|_{\mathfrak{x}}) \overset{(i)}{=} \Sigma(M_d\,P|_{\mathfrak{x}}) = \Sigma(N^{-1}P|_{\mathfrak{x}}) .$$

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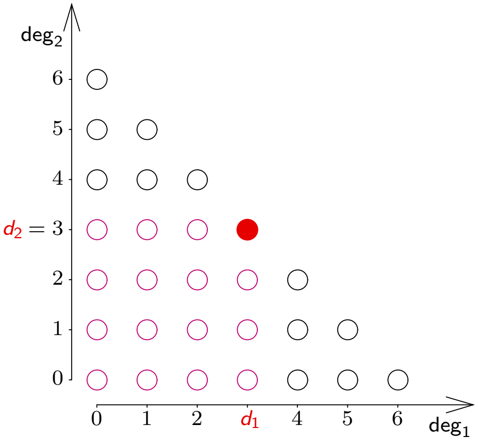
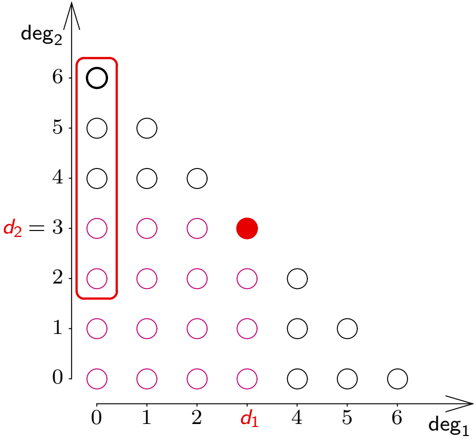
The transformation $P \longmapsto \longmapsto \cdots \longmapsto P/\mathfrak{X}$:



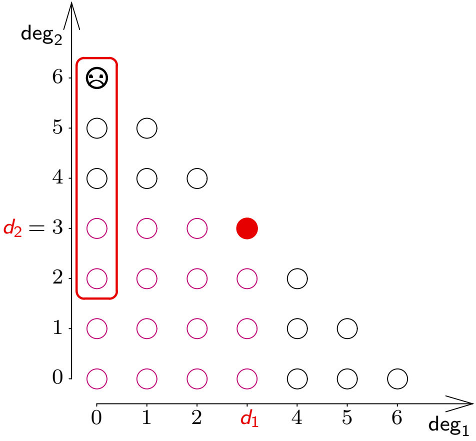
Start with P and add successively . . .

The transformation $P \longmapsto \longmapsto \cdots \longmapsto P/\mathfrak{X}$:



$$+ \quad c \;\; X^{(0,2)} \prod_{\xi \in \mathfrak{X}_2} (X_2 - \xi) \qquad \equiv 0 \;\text{ on } \mathfrak{X}$$

The transformation $P \longmapsto \longmapsto \cdots \longmapsto P/\mathfrak{X}$:



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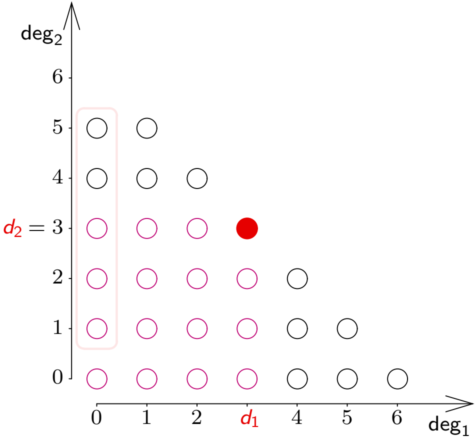

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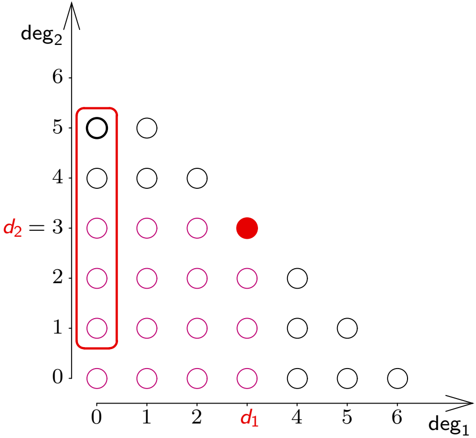

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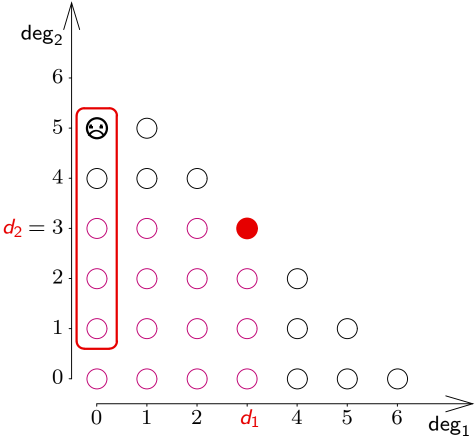

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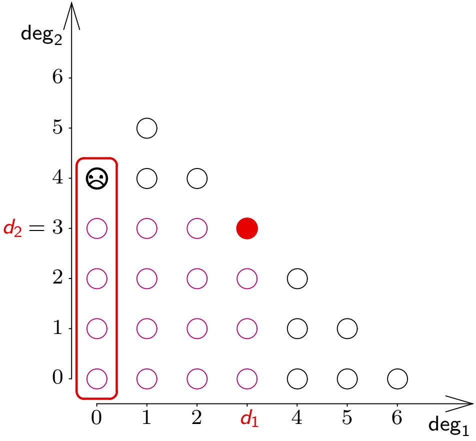

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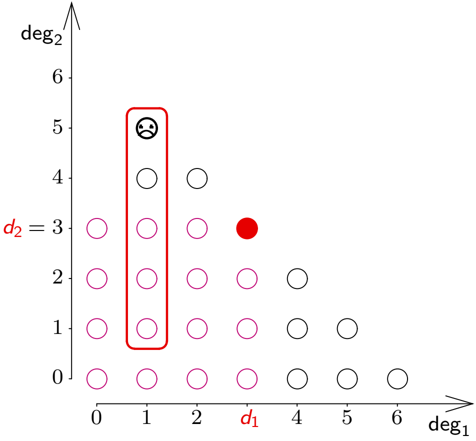

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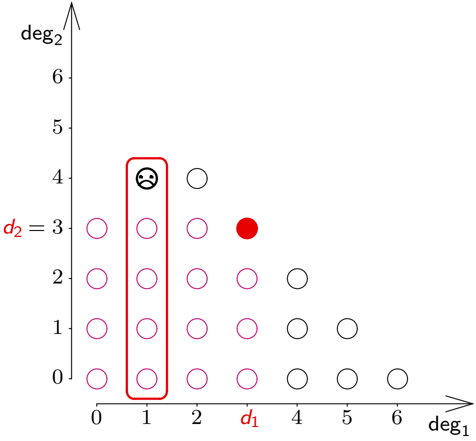

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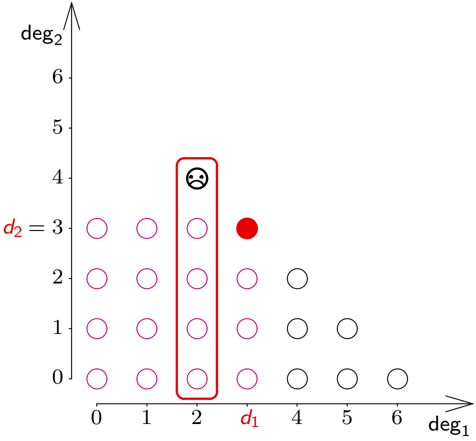

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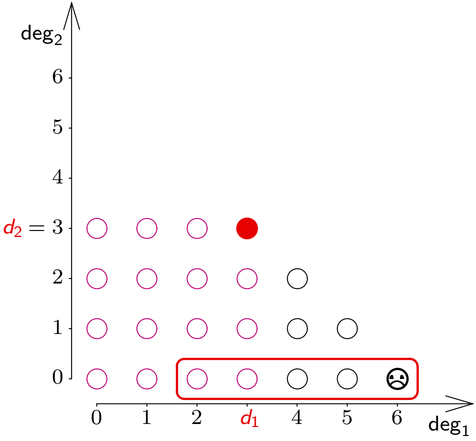

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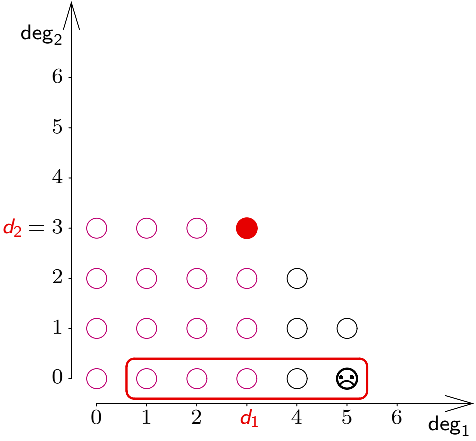

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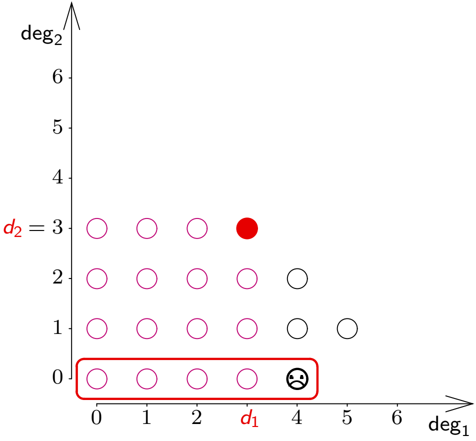

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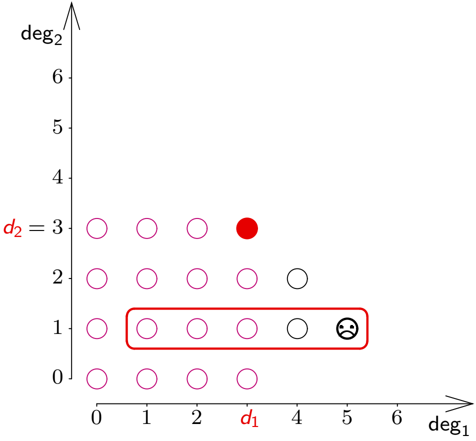

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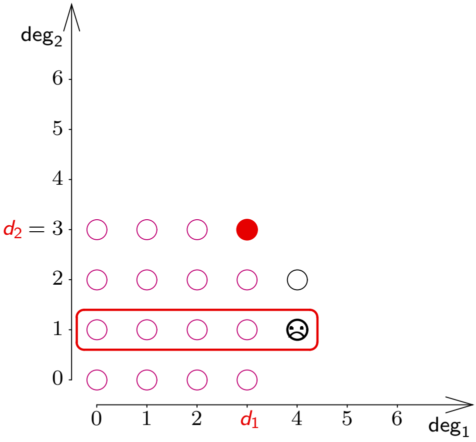

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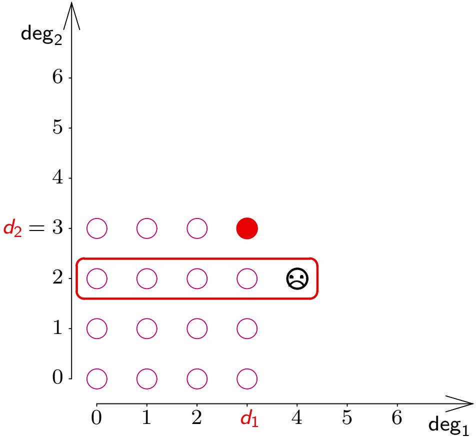

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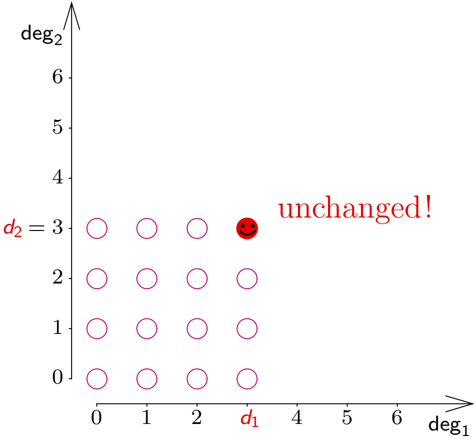

The trimmed polynomial $P/\mathfrak{X}$.

Specializations of the Coefficient Formula

If $\deg(P) \leq \Sigma d$, then
$$P_d = \sum_{x \in \mathfrak{X}} N(x)^{-1} P(x)$$

Let $A \in \mathcal{R}^{m \times n}$, $b \in \mathcal{R}^m$.
If $m \leq \Sigma d$, then
$$\operatorname{per}_d(A) = \sum_{x \in \mathfrak{X}} N(x)^{-1} \overbrace{\prod (Ax - b)}^{\text{Matrix Poly.}}$$

Let d_v denote the indegree of the vertices $v \in V$ of $\vec{G} = (V, \vec{E})$ and let $\mathfrak{X}_v \subseteq \mathcal{R}$ be a "list of $d_v + 1$ colors" so that the set $\mathfrak{X} := \prod_{v \in V} \mathfrak{X}_v$ of potential list colorings of $\vec{G}$ is a d-grid for $d := (d_v)_{v \in V}$, then
$$\pm \underbrace{|EE| \mp |EO|}_{\text{Eulerian Subgraphs}} = \operatorname{per}_d(\underbrace{A(\vec{G})}_{\text{Incidence Matrix}}) = \sum_{x \in \mathfrak{X}} N(x)^{-1} \overbrace{\prod_{\vec{st} \in \vec{E}} (x_t - x_s)}^{\text{Graph Poly.}}$$

If $\vec{L}$ is the arbitrarily oriented line graph of a planar k-regular graph G and $d_e = k - 1$ for all $e \in E(G) = V(\vec{L})$, then
$$\text{const} \cdot \operatorname{per}_d(A(\vec{L})) = \text{“the number of edge } k\text{-colorings of } G\text{”}$$

Describing Polynomials as Equivalent to Explicit Solutions

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King Fhad University of Petroleum and Minerals, Saudi Arabia

Thurnau, April 14, 2010

Doctoral Thesis: http://tobias-lib.ub.uni-tuebingen.de/volltexte/2007/2955

Applications

1. Alon and Tarsi's Combinatorial Nullstellensatz.
2. Chevalley and Warning's Theorem about the number of simultaneous zeros of systems of polynomials over finite fields. A sharpening of Warning's lower bound for this number and a generalization of Olson's version.
3. Ryser's Permanent Formula.
4. Alon's Permanent Lemma.
5. Alon and Tarsi's Theorem about orientations and colorings of graphs.
6. Scheim's formula for the number of edge n-colorings of planar n-regular graphs.
7. Ellingham and Goddyn's partial answer to the list coloring conjecture.
8. Alon, Friedland and Kalai's Theorem about regular subgraphs.
9. Alon and Füredi's Theorem about cube covers.
10. Cauchy and Davenport's Theorem from additive number theory.
11. Erdős, Ginzburg and Ziv's Theorem from additive number theory.

Definition (δ-permanent)

Let $A\langle|\delta\rangle$ be a matrix that contains the j^{th} column of A exactly δ_j times.

The *δ-permanent* of $A = (a_{i,j}) \in \mathcal{R}^{m\times n}$ is defined through

$$\mathrm{per}_\delta(A) := \sum_{\substack{\sigma:\ (m]\to(n] \\ |\sigma^{-1}(j)|=\delta_j}} \prod_{i=1}^{m} a_{i,\sigma(i)} = \begin{cases} \frac{1}{\prod(\delta_j!)}\,\mathrm{per}(A\langle|\delta\rangle) & \text{if } \Sigma\delta = m\,, \\ 0 & \text{else}\,. \end{cases}$$

Lemma (The Coefficients of the Matrix Polynomial)

$$\Pi(AX) = \sum_{\delta\in\mathbb{N}^n} \mathrm{per}_\delta(A)\, X^\delta\,,$$

$$\Pi(AX-b) := \prod_{i=1}^{m}\Big(\big(\sum_{j=1}^{n} a_{ij}X_j\big) - b_i\Big)$$

$$= \Pi(AX) + \text{"a polynomial of lower degree"}\,.$$

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Conjecture (Alon, Friedland, Kalai)

Set $\mathcal{R} := \mathbb{Z}/k\mathbb{Z}$, let $\mathfrak{X} := \{0,1\}^n$ be the Boolean grid and let $P_1, \dots, P_m \in \mathcal{R}[X_1, \dots, X_n]$ be homogenous polynomials of degree 1.

If $(k-1)\,m < n$, then there is a nontrivial simultaneous zero, i.e.,

$$\boxed{\left|\{\, x \in \mathfrak{X} \mid P_1(x) = \cdots = P_m(x) = 0 \,\}\right| \neq 1}\,.$$

Theorem (Generalized Olson-Theorem)

Let $p \in \mathbb{N}$ be a prime and $\mathfrak{X} \subseteq \mathbb{Z}^n$ a d-grid with the additional property that for all $j \in \{1, \dots, n\}$ and all $x, \tilde{x} \in \mathfrak{X}_j$ with $x \neq \tilde{x}$ holds $p \nmid x - \tilde{x}$. For polynomials $P_1, \dots, P_m \in \mathbb{Z}[X_1, \dots, X_n]$, and numbers $k_1, \dots, k_m > 0$ small enough so that $\sum_i (p^{k_i} - 1)\deg(P_i) < \Sigma d$,

$$\boxed{\left|\{\, x \in \mathfrak{X} \mid \forall i\colon\ p^{k_i} \mid P_i(x) \,\}\right| \neq 1}\,.$$

Conjecture (Alon, Friedland, Kalai)

Set $\mathcal{R} := \mathbb{Z}/k\mathbb{Z}$, let $\mathfrak{X} := \{0,1\}^n$ be the Boolean grid and let $P_1,\dots,P_m \in \mathcal{R}[X_1,\dots,X_n]$ be homogenous polynomials of degree 1.

If $(k-1)\,m < n$, then there is a nontrivial simultaneous zero, i.e.,

$$\boxed{\left|\{\, x \in \mathfrak{X} \;\mid\; P_1(x) = \dots = P_m(x) = 0 \,\}\right| \neq 1}\,.$$

Theorem (Generalized Olson-Theorem)

Let $p \in \mathbb{N}$ *be a prime and* $\mathfrak{X} \subseteq \mathbb{Z}^n$ *a d-grid with the additional property that for all* $j \in \{1,\dots,n\}$ *and all* $x, \tilde{x} \in \mathfrak{X}_j$ *with* $x \neq \tilde{x}$ *holds* $p \nmid x - \tilde{x}$. *For polynomials* $P_1,\dots,P_m \in \mathbb{Z}[X_1,\dots,X_n]$, *and numbers* $k_1,\dots,k_m > 0$ *small enough so that* $\sum_i (p^{k_i}-1)\deg(P_i) < \Sigma d$,

$$\boxed{\left|\{\, x \in \mathfrak{X} \;\mid\; \forall\, i\colon\; p^{k_i} \mid P_i(x) \,\}\right| \neq 1}\,.$$