Resolvable Steiner 3-Designs

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Kirkman: Resolvable 2-$(15, 3, 1)$ Steiner design.

15 young ladies in a school walk out three abreast for 7 days in succession; it is required to arrange them daily, so that no two shall walk twice abreast.

$\exists$ **resolvable** t-designs for $t > 2$ large k [3], but $\lambda > 1$.

M. Sawa:
$\exists$ resolvable **Steiner** t-designs, i.e. $\lambda = 1$, for $t > 2$ large k?
$\longrightarrow$ construction of infinite families of 3-designs [4].

Hartman and Ji,Zhu [1, 2]:

$$\exists \text{ resolvable } 3\text{-}(v, 4, 1) \text{ design}$$

$$\Longleftrightarrow$$

$$v \equiv 4,\ 8 \ mod\ 12$$

$k > 4 :\quad 5\text{-}(12, 6, 1),\ 5\text{-}(24, 8, 1),\ 5\text{-}(48, 6, 1),\ ..?$

Theorem

Let q prime power:

$$\exists \textit{ resolvable } 3\text{-}(q^n+1, q+1, 1) \textit{ design}$$

$$\Longleftrightarrow$$

$$q+1 \mid q^n+1, \;\; \textit{i.e. } n \textit{ odd}$$

Proof by help of good friends: groups

$\exists$ 3-$(q^n+1, q+1, 1)$ design $\mathcal{D}$,
with group of automorphisms $G = PGL(2, q^n)$
n even: $\Longrightarrow (q+1) \not| (q^n+1)$.

n odd: **Claim: this design $\mathcal{D}$ is resolvable**

$G = PGL(2, q^n)$ is 3-homogeneous $\Longrightarrow$ Any $(q+1)$-orbit is a 3-design.

$B = PG(1, \mathbb{F}_q) < PG(1, \mathbb{F}_{q^n})$ is $(q+1)$-set, B^G block set of a 3-design $\mathcal{D}$.

B orbit of $PGL(2, q) \leq G_B < PGL(2, q^n)$

$$\frac{(q^n+1)q^n(q^n-1)}{(q+1)q(q-1)}\lambda = |B^G| = \frac{|G|}{|G_B|}$$

divides

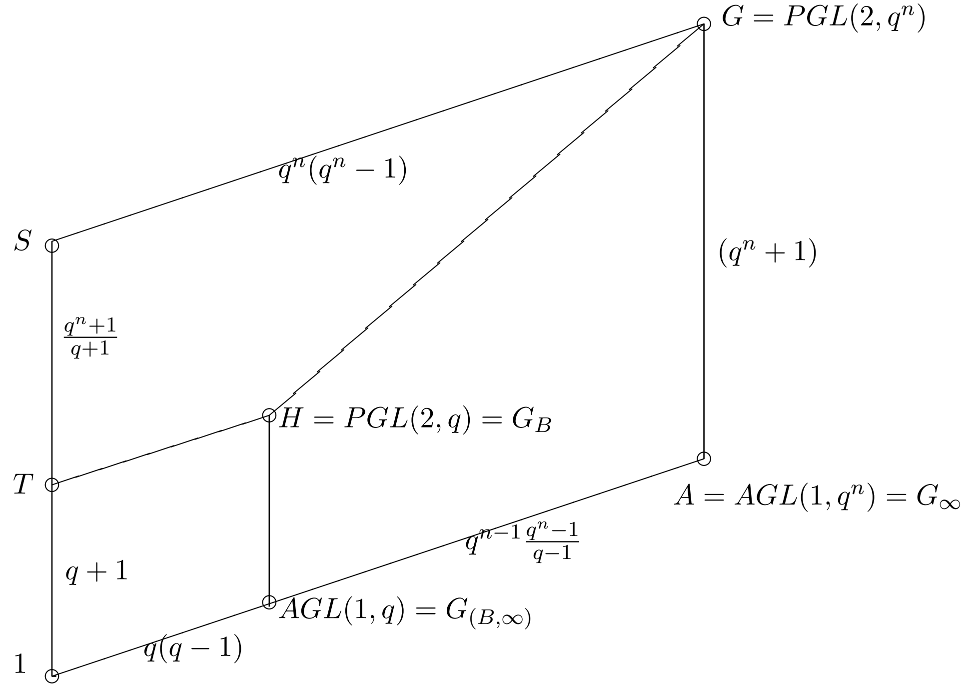
$$\frac{|PGL(2, q^n)|}{|PGL(2, q)|} = \frac{(q^n+1)q^n(q^n-1)}{(q+1)q(q-1)}$$

$$\Longrightarrow \lambda = 1, \ G_B = PGL(2, q)$$

$A = G_\infty \cong AGL(1, \mathbb{F}_{q^n})$, $\infty \in B$, $H = G_B$

S Singer of G , $T = S \cap H$ Singer of H.

B is T-orbit, B^S partitions $PG(1, \mathbb{F}_{q^n})$ into blocks.



Double cosets correspond to splitting of an orbit.

$$\infty^G/H \cong A\backslash G/H$$

$$B^G/A \cong H\backslash G/A.$$

$$\mathbf{A\backslash G/H} \longleftrightarrow \mathbf{H\backslash G/A}$$
$$AgH \longleftrightarrow Hg^{-1}A$$
$$|AgH| = |Hg^{-1}A|$$

H fixed point freely on $X = PG(1, \mathbb{F}_{q^n}) \setminus B$:

orbits of H on $PG(1, \mathbb{F}_{q^n})$:

1 orbit B of size $q + 1$,

s **regular** orbits of size $(q + 1)q(q - 1)$.

orbits of A on B^G:

$der_\infty(\mathcal{D})$ containing B, $A_B = A \cap H$

s **regular** orbits of size $q^n(q^n - 1)$ on $res_\infty(\mathcal{D})$.

$$B',\ T\text{-orbit},\ B' \neq B,\ B' \subseteq X'\ H\text{-orbit}$$

$$\Longrightarrow$$

$$B'^{H \cap A} = \text{ is a partition of } X'$$

Choose T-orbit B_i within each H orbit on X.

$$P = (B, B_i^{H \cap A} | 1 \leq i \leq s) \text{ partition of } PG(1, \mathbb{F}_{q^n})$$

P^A **resolution** of 3-$(q^n+1, q+1, 1)$ design $\mathcal{D}$:

$der_\infty \mathcal{D}$	$res_\infty \mathcal{D}$	P^A
$q+1$	s times $(q+1)q(q-1)$	
$\infty \in B$	. … . . … . … . … .	P
	. … . . … . … . … .	.
…	…	
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	. … . . … . … . … .	.
A-orbit	s A-orbits of H-orbits	

Each block appears exactly once:

$$A \text{ has regular orbits on } res_\infty(\mathcal{D})$$

$$\{id\} = A_{B'} < H \cap A < A$$
$$\Longrightarrow$$

$$B'^{H\cap A} \text{ block of imprimitivity for } A$$

Wielandt $\Longrightarrow$ B'^A decomposes into disjoint blocks of imprimitivity.

B_i orbit of T on H-orbit X_i.

$B_i = B^{g_i}$ lies in the A-orbit $B^{g_i A}$ that corresponds to the double coset Hg_iA.

If $B_i^A = B_j^A$ and $i \neq j$ then $Hg_iA = Hg_jA$ and $Ag_j^{-1}H = Ag_i^{-1}H$.

Then $\infty^{g_j^{-1}H} = \infty^{g_i^{-1}H}$. But B_i and B_j were selected from different H-orbits.

Thus, the orbits B_i^A are disjoint regular orbits of A on the block set of $res_\infty(\mathcal{D})$.

Since the number of blocks of the residual design is just $s \cdot |A|$, we have obtained a resolution.

65 young ladies in a school walk out five abreast for 336 days in succession; it is required to arrange them daily, so that no three shall walk twice abreast.

2198 young ladies in a school walk out 14 abreast for 30927 days, that is 84 years and several days, in succession; it is required to arrange them daily, so that no three shall walk twice abreast.

References

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[3] R. Laue, Resolvable t-designs, *Des. Codes Cryptogr.* **32** (2004), 277–301.

[4] M. Jimbo, Y. Kunihara, M. Sawa, R. Laue, Unifying some known infinite families of combinatorial 3-designs, in preparation.