A Census of One-Factorizations of the Complete 3-Uniform Hypergraph of Order 9

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Joint work with:

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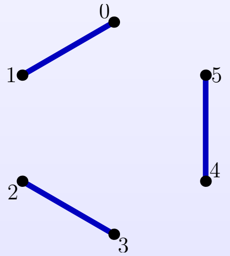

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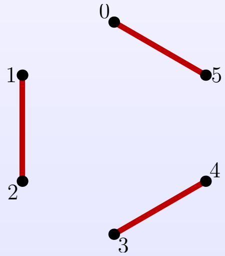

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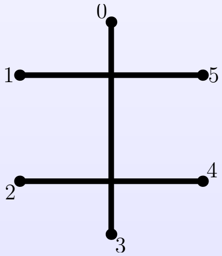

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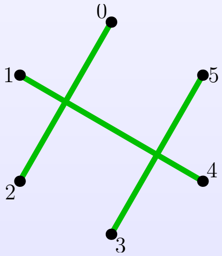

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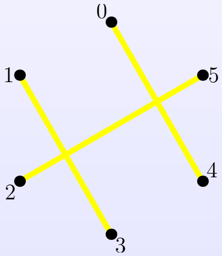

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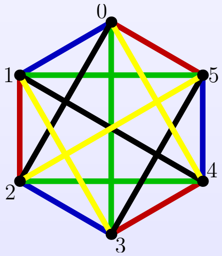

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$$E(K_6^3) = \{ \{0,1,2\},\{3,4,5\}, \{0,1,3\},\{2,4,5\}, \ldots \}$$

$\therefore K_6^3$ has a unique one-factorization.

Theorem (Baranyai, 1975)

K_n^k has a one-factorization if only if $k|n$.

$$N(K_n^k) = \text{Number of nonisomorphic one-factorizations of } K_n^k$$

n	2	4	6	8	10	12	14
$N(K_n^2)$	1	1	1	6	396	526915620[DGM]	1132835421602062347[KÖ]

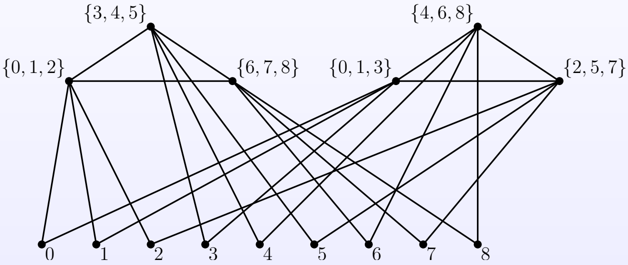
n	3	6	9
$N(K_n^3)$	1	1	?[MR]

[DGM]: Dinitz, Garnick, McKay (1994)

[KÖ]: Kaski, Östergård (2009)

[MR]: Mathon, Rosa (1983): K_9^3 has 130 one-factorizations with automorphism group of order > 4

$\mathcal{F}$: a set of disjoint one-factors $\xrightarrow{\textit{associate}}$ $G(\mathcal{F})$: a graph



Proposition

Two sets of one-factors $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic if and only if the graphs $G(\mathcal{F}_1)$ and $G(\mathcal{F}_2)$ are isomorphic.

- K_9^3 has $\binom{9}{3}$ edges.
- K_9^3 has $\binom{9}{3}\binom{6}{3}/3! = 280$ one-factors.
- A one-factorization of K_9^3 has $\binom{9}{3}/3 = 28$ one-factors.

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So, we need to decide how many sets out of the $\binom{280}{28} \approx 3 \times 10^{38}$ sets form a one-factorization (up to isomorphism).

We may assume that the edges $\{0,1,i\}$, $2 \le i \le 8$, belong to the first seven one-factors.

Definition

Seed: a set of seven one-factors $\{F_1, \ldots, F_7\}$ so that there exist $0 \le a < b \le 8$ such that

$$\{\{a,b,i\} : 0 \le i \le 8, i \ne a,b\} \subset \bigcup_{j=1}^{7} F_j.$$

Every one-factorization contains exactly $\binom{9}{2} = 36$ seeds.

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$$\{\{0,1,3\},\{2,4,8\},\{5,6,7\}\} \text{ (1st choice)}$$

$$\{\{0,1,3\},\{2,6,8\},\{4,5,7\}\} \text{ (2nd choice)}$$

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A backtrack search, adding one-factors that contain an edge of the form $\{0, 1, i\}$ one at a time and carrying out isomorph rejection.

The *nauty* library by McKay is used to handle the $G(\mathcal{F})$ graphs.

Table: Number of partial seeds

# of one-factors in a partial seed	1	2	3	4	5	6	7
# of partial seeds	1	2	11	45	156	277	208

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There are 208 non-isomorphic seeds.

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The instances of finding one-factorizations from the given seeds lead to a total of 8 185 376 solutions.

Methods available:

(1) Recorded objects

(2) Canonical augmentation (by McKay, 1998)

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(ii) Identify a canonical $\mathrm{Aut}(\mathcal{F})$-orbit of seeds contained by $\mathcal{F}$, and then check whether the seed from which $\mathcal{F}$ was extended is in the canonical orbit.

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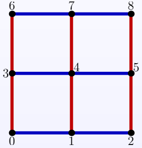
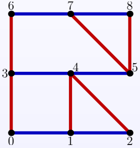
F, G: two one-factors of K_9^3



$\alpha(F, G) = 9$



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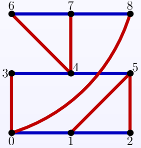

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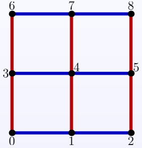

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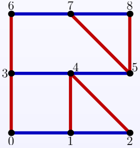

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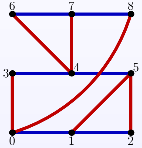

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$\mathcal{S} = \{F_1, \dots, F_7\}$: a seed $\xrightarrow{\textit{associate}}$ $d(\mathcal{S}) = \{d(F_1), \dots, d(F_7)\}$: an invariant, where

$$d(F_j) = \sum_{i \neq j} \alpha(F_i, F_j),$$

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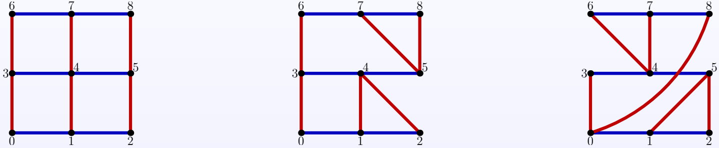

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Speed up:

The canonical $\mathrm{Aut}(\mathcal{F})$-orbit of seeds in (ii) is required to have the lexicographically smallest invariant $d(\mathcal{S})$.

There are exactly 103 000 isomorphism classes of one-factorizations of K_9^3.

$\lvert\mathrm{Aut}(\mathcal{F})\rvert$	#	$\lvert\mathrm{Aut}(\mathcal{F})\rvert$	#
1	99 453	16	2
2	3 151	18	3
3	151	24	5
4	111	36	1
6	84	42	1
7	2	54	2
8	10	56	1
9	1	336	1
12	17	432	1
14	2	1 512	1

A consistency check

During the main search, we record

(i) $|\mathrm{Aut}(\mathcal{S}_i)|$: for each seed $\mathcal{S}_i$,

(ii) M_i: the total number of one-factorizations found by the exact cover algorithm as extensions of $\mathcal{S}_i$, and

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By the orbit-stabilizer theorem, the total # of one-factorizations of K_9^3:

$$\frac{1}{\binom{9}{2}} \sum_{i=1}^{208} \frac{9! \cdot M_i}{|\mathrm{Aut}(\mathcal{S}_i)|} = \sum_{i=1}^{103\,000} \frac{9!}{|\mathrm{Aut}(\mathcal{F}_i)|}.$$

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Both the sides evaluate to 36 696 023 040.

N partial seeds $\mathcal{F}_1, \ldots, \mathcal{F}_N$ with $|\mathcal{F}_i| = m - 1$

$\hookrightarrow$ N' partial seeds $\mathcal{F}'_1, \ldots, \mathcal{F}'_{N'}$ with $|\mathcal{F}'_i| = m$

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The total number of partial seeds of size m (by the orbit-stabilizer thm):

$$\frac{1}{m} \sum_{i=1}^{N} \frac{7! \cdot M_i}{|\mathrm{Aut}(\mathcal{F}_i)|} = \sum_{i=1}^{N'} \frac{7!}{|\mathrm{Aut}(\mathcal{F}'_i)|}.$$

A consistency check

Validating the classification of seeds

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For $m = 1, \ldots, 7$ both sides evaluate to 70, 1 890, 25 410, 182 910, 701 820, 1 323 420, and 942 900, respectively.