Network Codes and Designs over Finite Fields

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- Designs
- Network Codes
- Construction

Designs

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- a set of v points

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- a set of blocks (block = set of points)

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  each block is a $k-$set
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$2-(7,3,1)$ design

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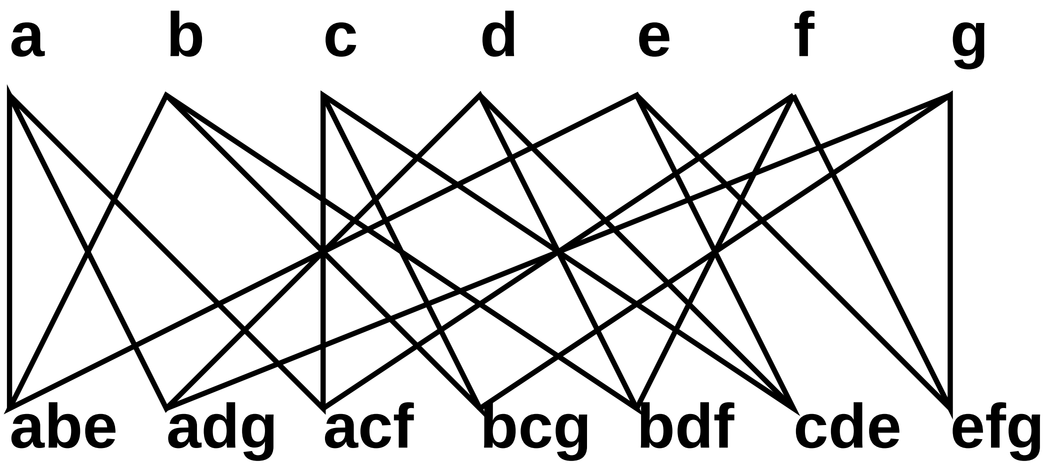
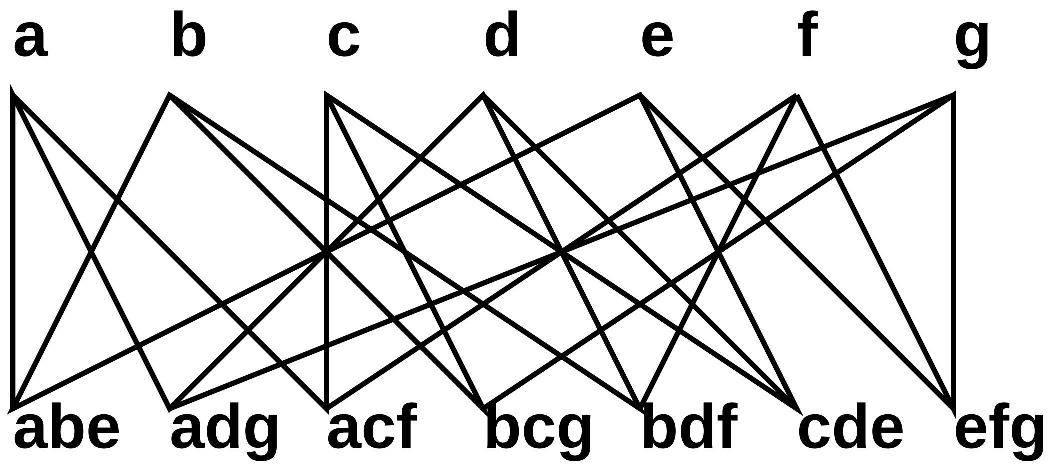
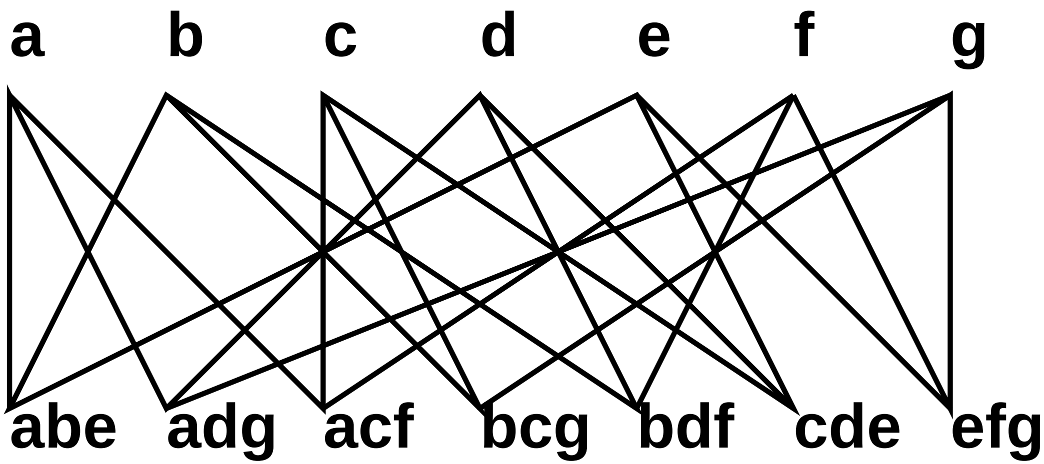
Designs



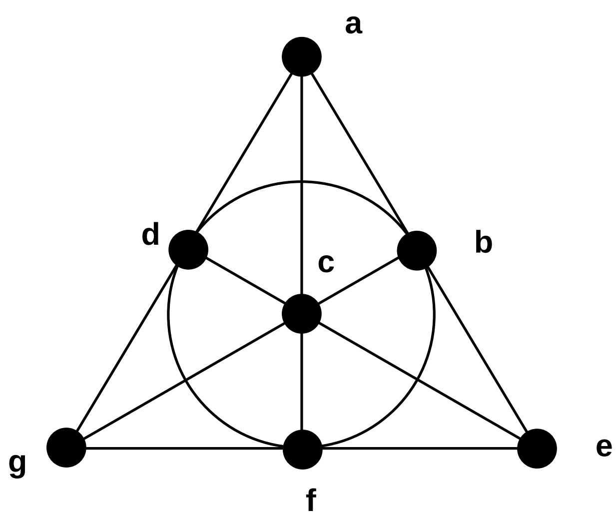
Heawood graph



Heawood graph



Fano plane



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- a set of $k-$blocks
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Designs over Finite Fields

- ~~a set of v points~~

  linear $v-$space $GF(q)^v$
- a set of $k-$blocks

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Designs over Finite Fields

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  a set of $k-$spaces in $GF(q)^v$

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- ~~$t-(v,k,\lambda)$ Design~~

  ~~each $t-$set of points is in exactly λ blocks~~

  $t-(v,k,\lambda)$ $q-$Design

  each $t-$space of $GF(q)^v$ is in exactly λ of the $k-$spaces

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known:

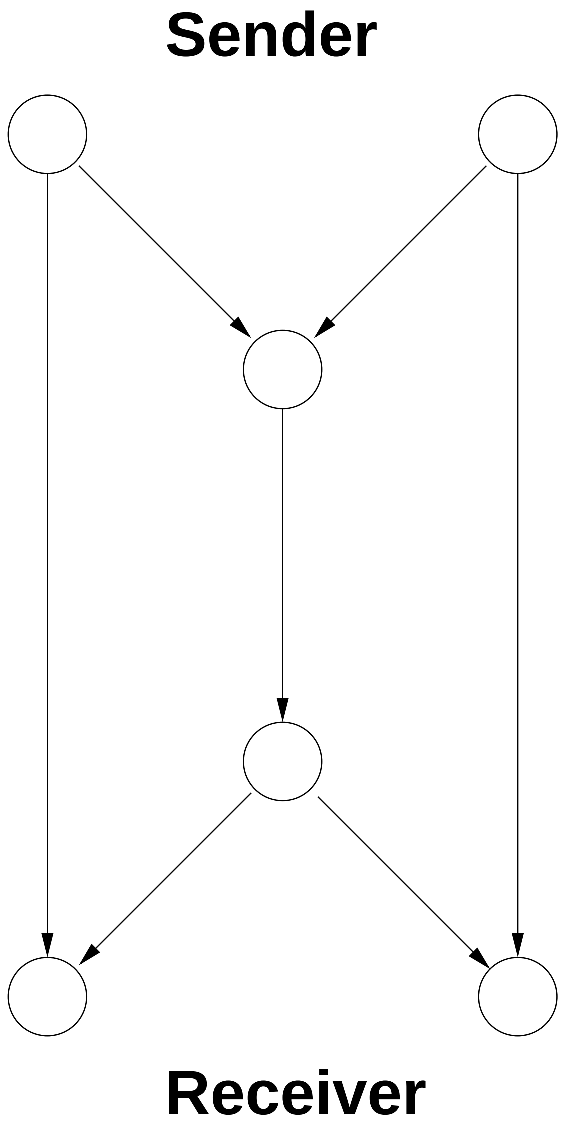
- Thomas (1987): first to study, $2-$designs
- Braun, Kerber, Laue (2005): first $3-$design

open problems:

- $q-$analog of the Fano plane?
- Steiner systems ? $(\lambda = 1)$
- $t > 3$?

Network Codes

Network Codes



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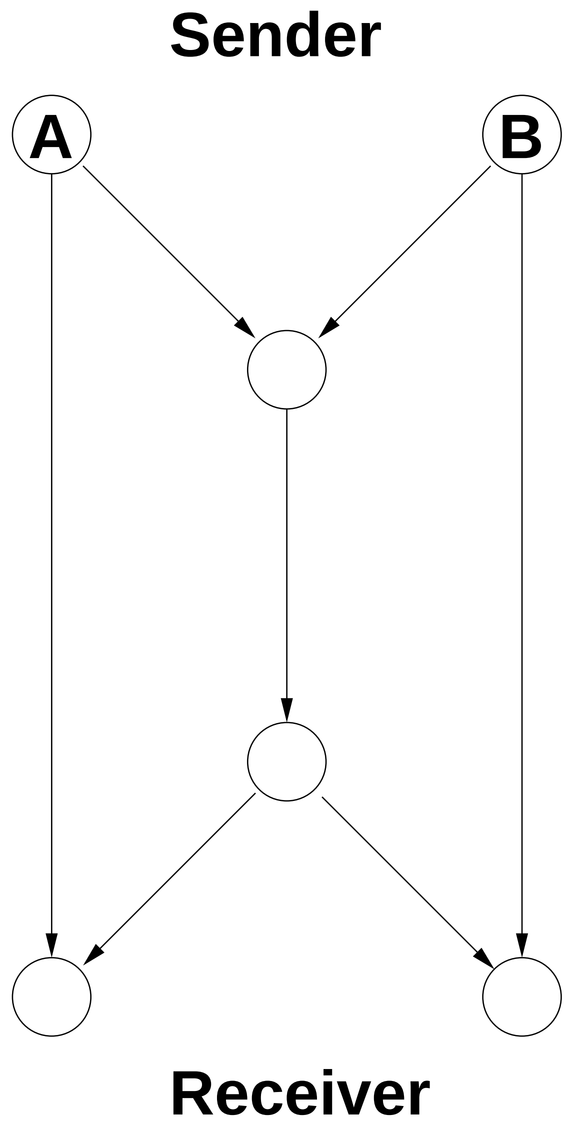
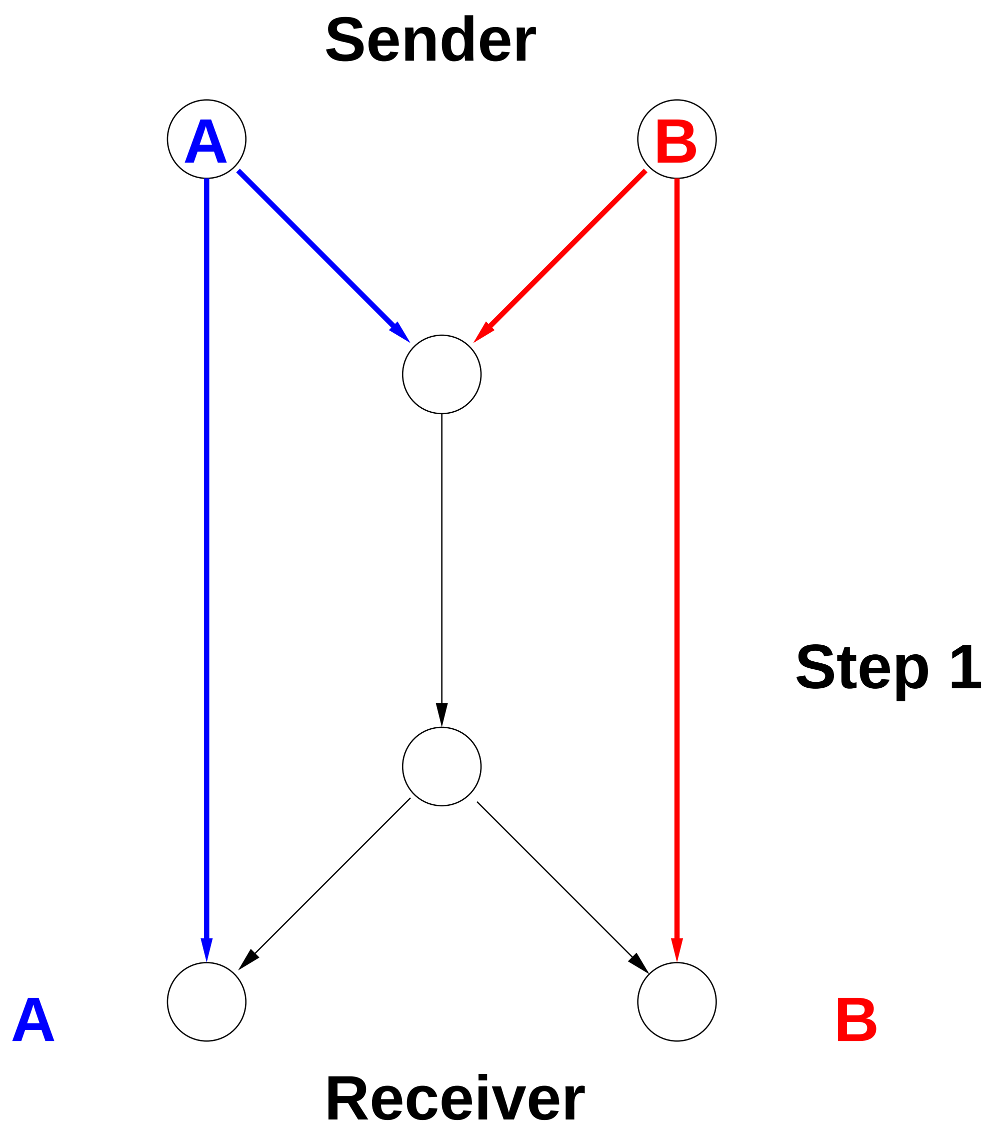
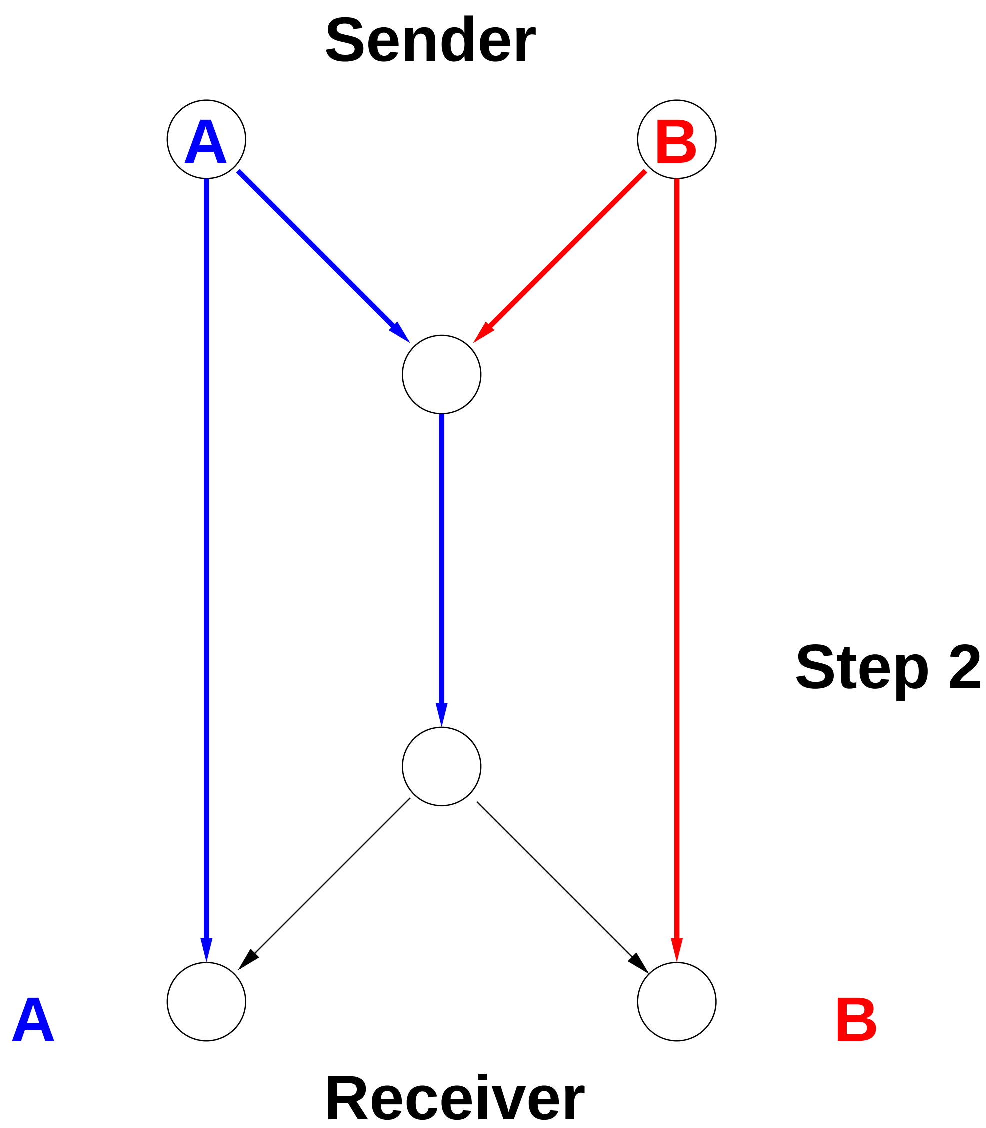
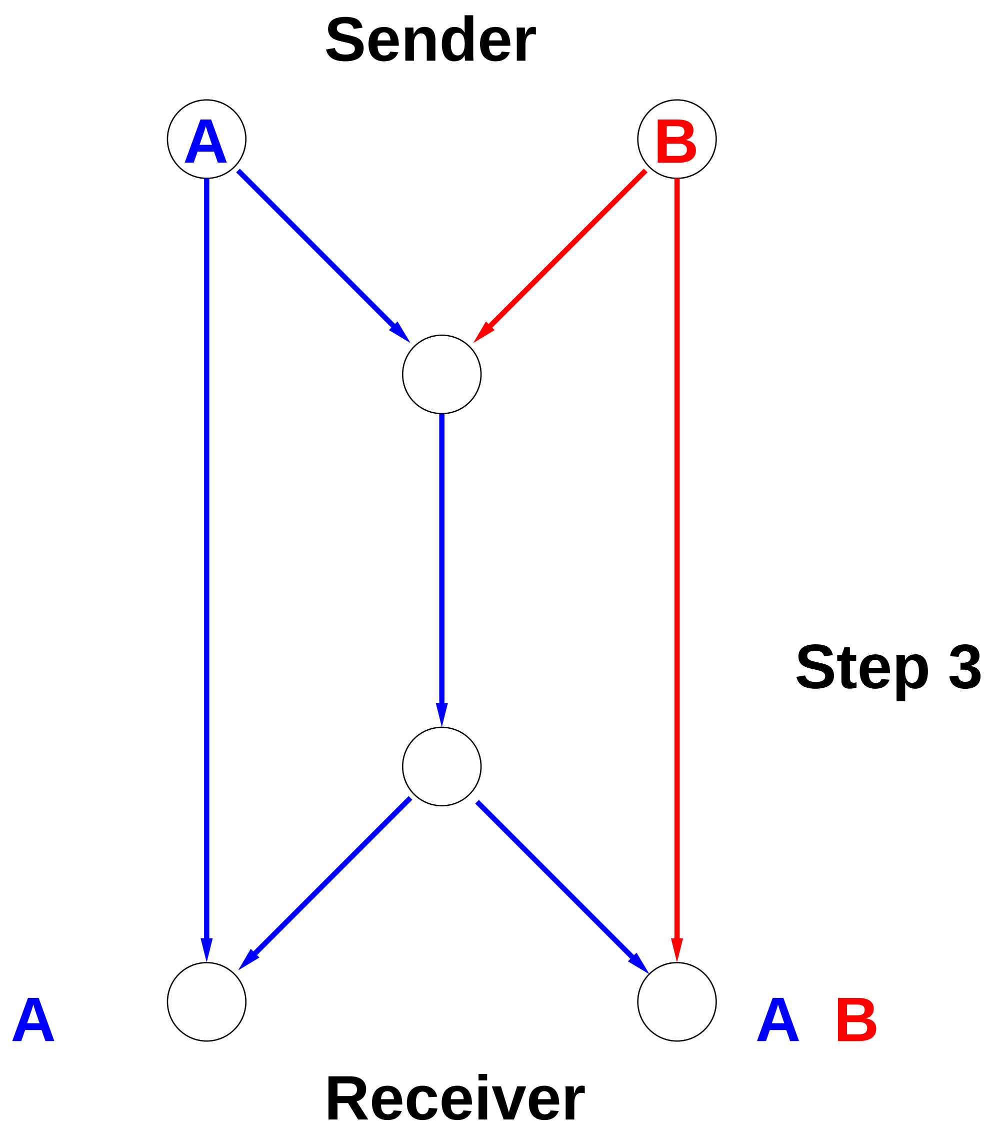
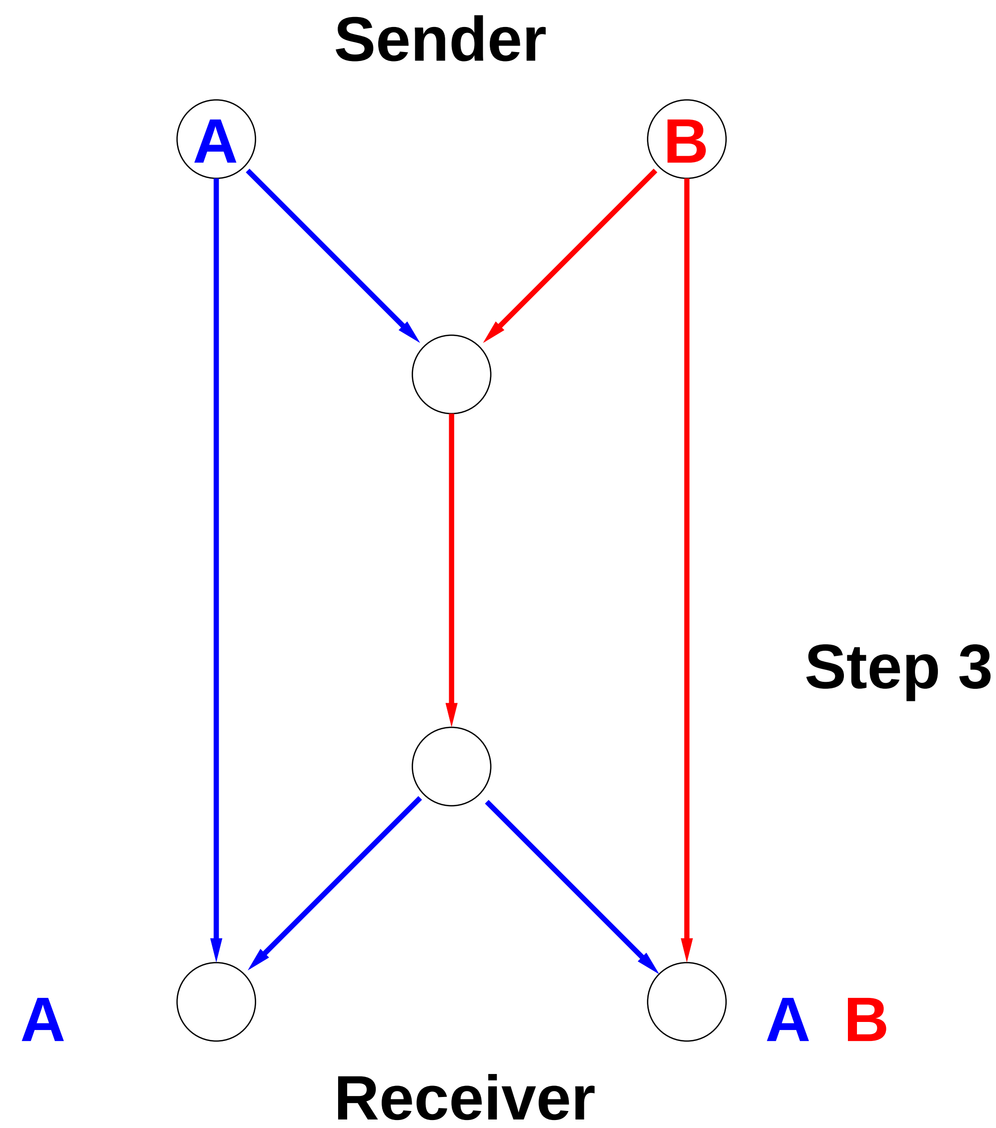
Network Codes

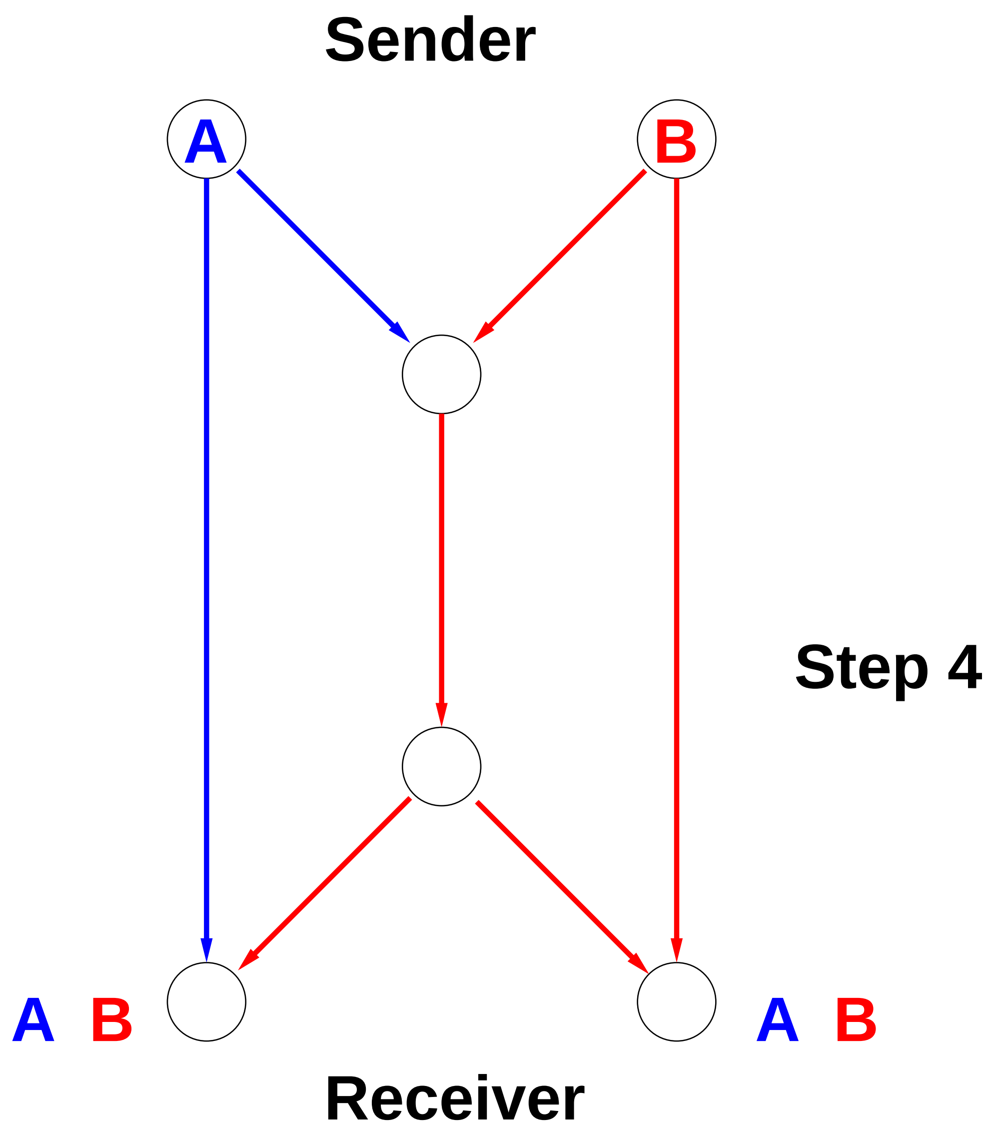


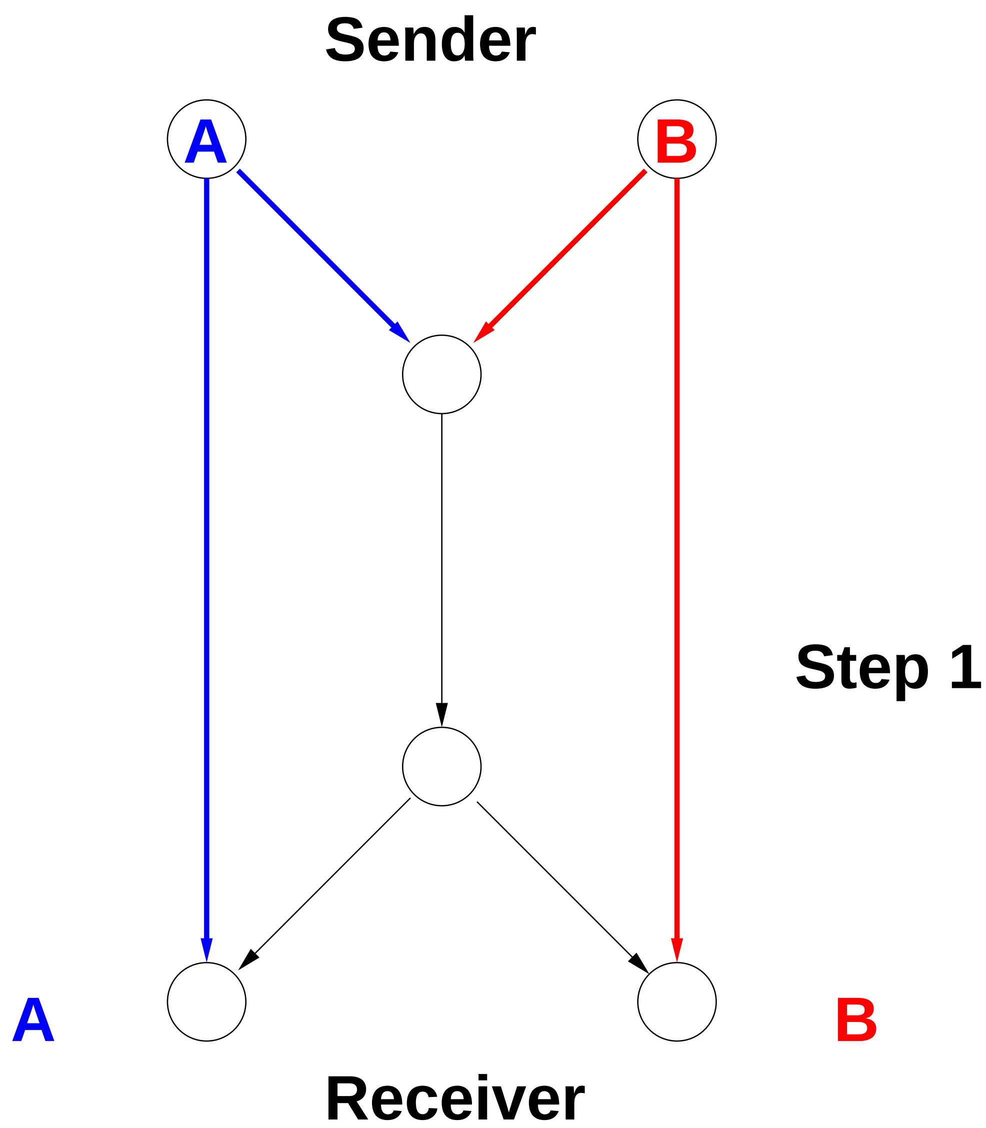

Sender
A
B
Step 1
A
B
Receiver

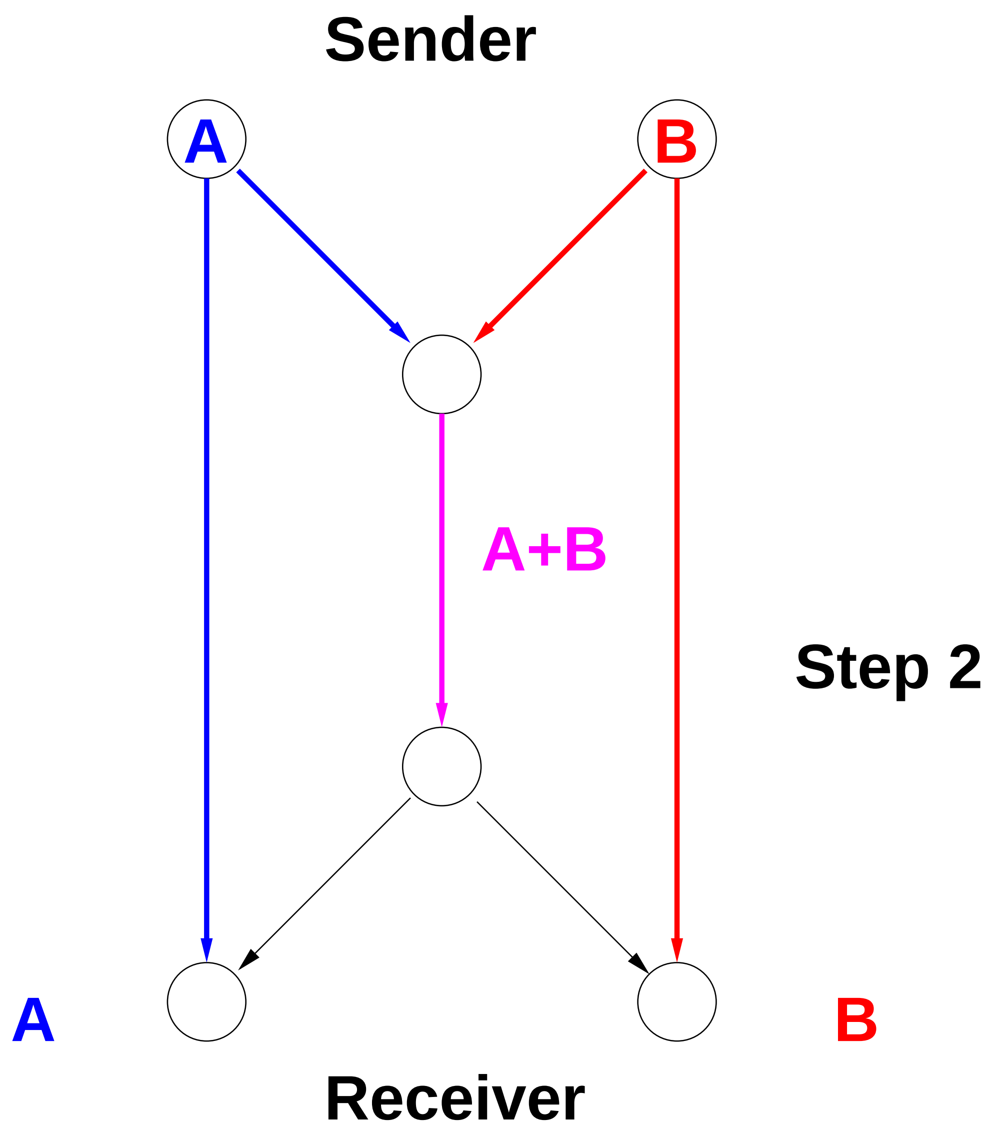

Sender
A
B
Step 2
A
B
Receiver

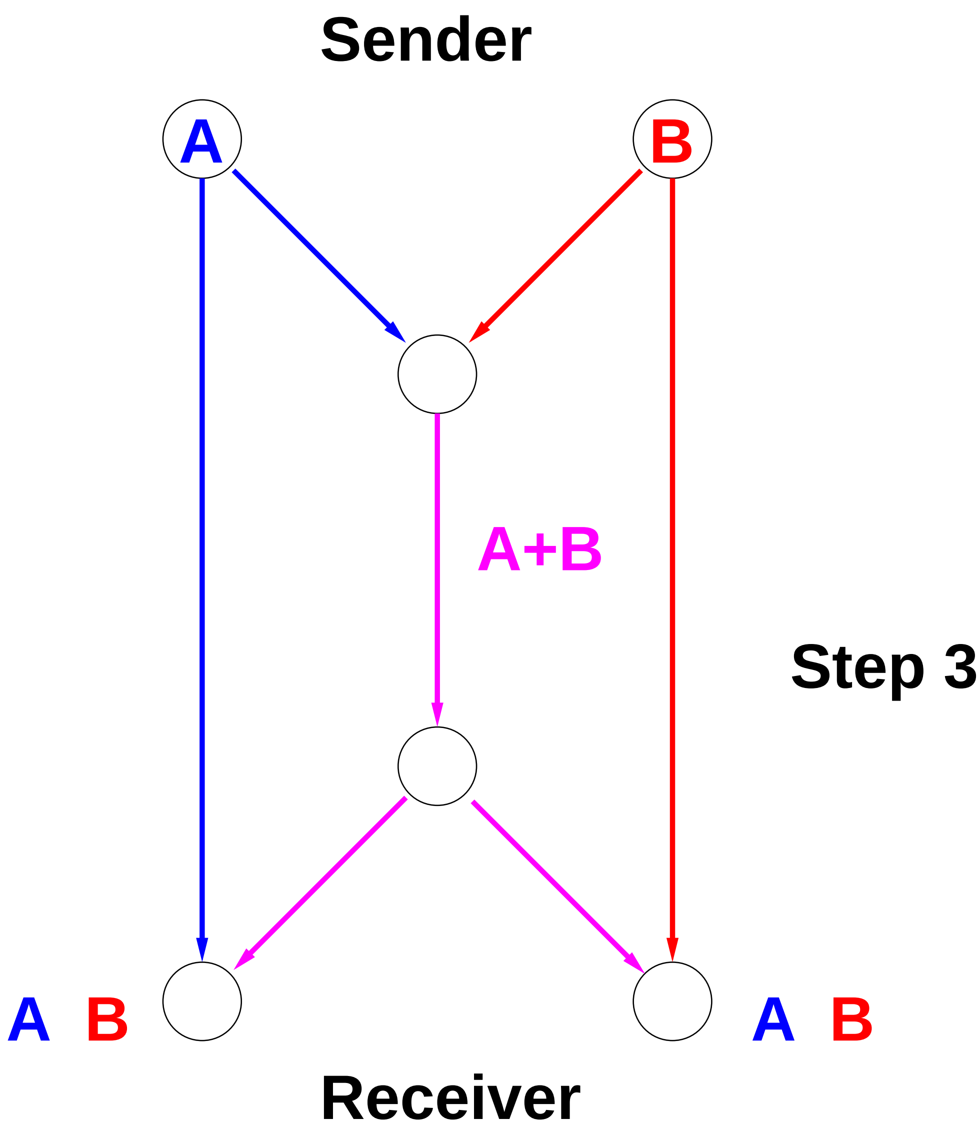

Sender
A
B
Step 3
A
A B
Receiver


Sender
A
B
Step 3
A
A B
Receiver


Sender
A
B
Step 4
A B
A B
Receiver


Sender
A
B
Step 1
A
B
Receiver


Sender
A
B
A+B
Step 2
A
B
Receiver


Sender
A
B
A+B
Step 3
A B
A B
Receiver

message:

- linear space

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single node:

- receives vectors
- sends some linear combination of the incoming vectors

Error-Correcting Network Codes

codeword:

- linear subspace of $GF(q)^v$

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U, W subspace of $GF(q)^v$:

$$d(U,W) = dim(U) + dim(W) - 2dim(U \cap W)$$

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fix minimum distance d:

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constant dimension codes $\approx$ constant weight codes

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Codes and Designs

Given a $t-(v,k,1)$ $q-$design we get a constant dimension code with minimum distance $2(k-(t-1))$ as the intersection of two codewords has dimension $\leq t-1$.

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= error-correcting network code

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Current State

Define $A_q(v,k,d)$ as the maximal size (= number of codewords) of a constant dimension code with minimum distance d, dimension of codewords = k, and ambient space = $GF(q)^v$

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open problems:

- find lower and upper bounds for $A_q(v, k, d)$
- find constructions of ’good’ codes
- special case $A_2(7, 3, 4)$ = Fano plane

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Construction

Problem

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in at most 1 $k-$subspace
= error-correcting network code

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D:= incidence matrix between $k-$spaces and $t-$spaces in $GF(q)^v$

$$D_{U,V} := \begin{cases} 1 & t\text{-space } U \text{ is subspace of } k - \text{space } W \\ 0 & \text{else} \end{cases}$$

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solution = network code with minimum distance $2(k - t + 1)$.

Automorphisms

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- D^G :=shrinked matrix

$\Rightarrow$number of columns = number of orbits on $k-$spaces
number of rows = number of orbits on $t-$spaces

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solution = network code with prescribed automorphisms and minimum distance $2(k - t + 1)$.

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Results (binary)

v	k	number of codewords: new	old	d
6	3	77	71	4
7	3	304	294	4
8	3	1275	1164	4
9	3	5621	4657	4
10	3	21483	18631	4
11	3	79833	74531	4
12	3	315315	298139	4

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A. Kohnert, S. Kurz: *Construction of Large Constant Dimension Codes With a Prescribed Minimum Distance*, LNCS, 2008.

R. Kötter, F. Kschischang: *Coding for errors and erasures in random network coding*, IEEE Transactions on Information Theory, **54**, 3579–3590, 2008.

Thank you very much for your attention.