Construction of Large Constant Dimension Codes with a Prescribed Minimum Distance

Axel Kohnert

Karlsruhe December 18, 2008

Bayreuth University Germany
axel.kohnert@uni-bayreuth.de

UNIVERSITÄT BAYREUTH

- Designs
- Network Codes
- Construction

Designs

Designs

- a set of v points

Designs

- a set of v points
- a set of blocks (block = set of points)

UNIVERSITÄT BAYREUTH

Designs

- a set of v points
- a set of blocks (block = set of points)
- $t - (v, k, \lambda)$ Design

UNIVERSITÄT BAYREUTH

Designs

- a set of v points
- a set of blocks (block = set of points)
- $t-(v,k,\lambda)$ Design
  each block is a $k-$set
  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs

- a set of v points

  a, b, c, d, e, f, g

- a set of blocks (block = set of points)

- $t-(v,k,\lambda)$ Design

  each block is a $k-$set

  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs

- a set of v points

  a, b, c, d, e, f, g
- a set of blocks (block = set of points)

  $abe, adg, acf, bcg, bdf, cde, efg$
- $t-(v,k,\lambda)$ Design

  each block is a $k-$set

  each $t-$set of points is in exactly λ blocks

Designs

- a set of v points

  a, b, c, d, e, f, g

- a set of blocks (block = set of points)

  $abe, adg, acf, bcg, bdf, cde, efg$

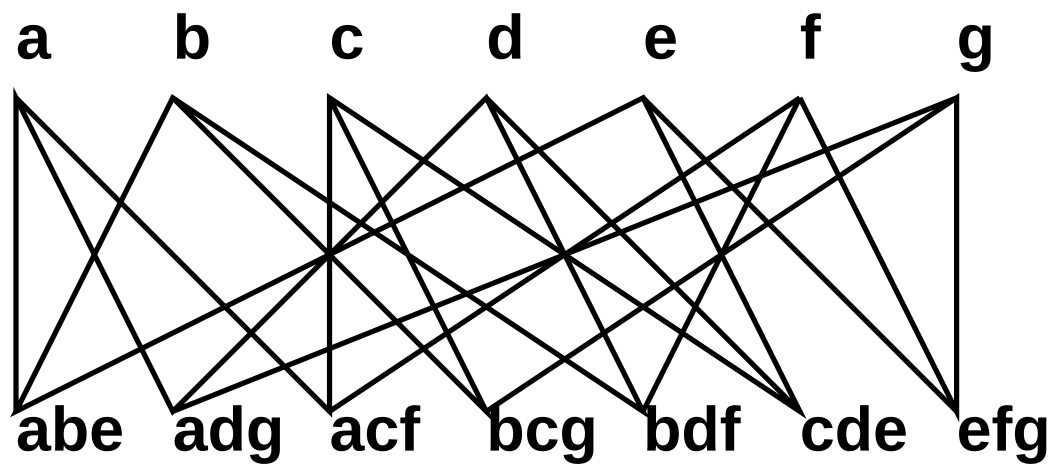
- $t-(v,k,\lambda)$ Design

  each block is a $k-$set

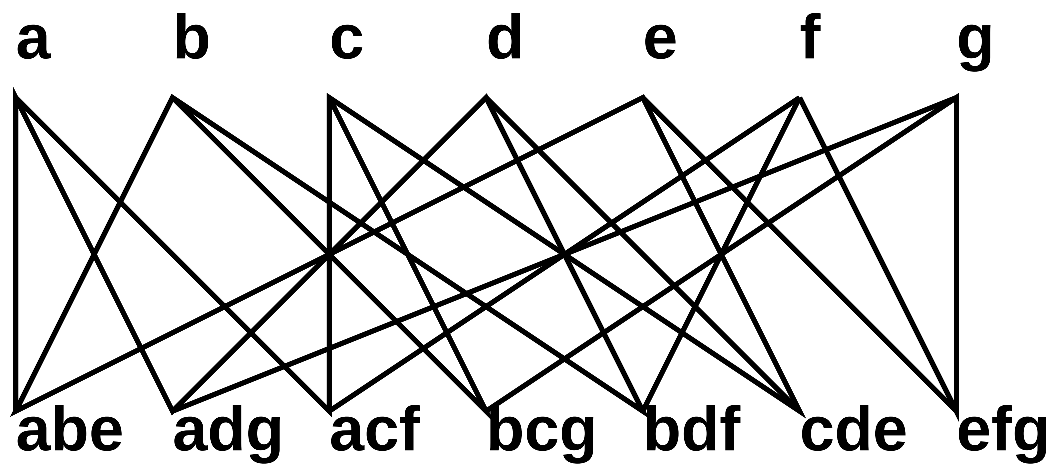
  each $t-$set of points is in exactly λ blocks

  $2-(7,3,1)$ design

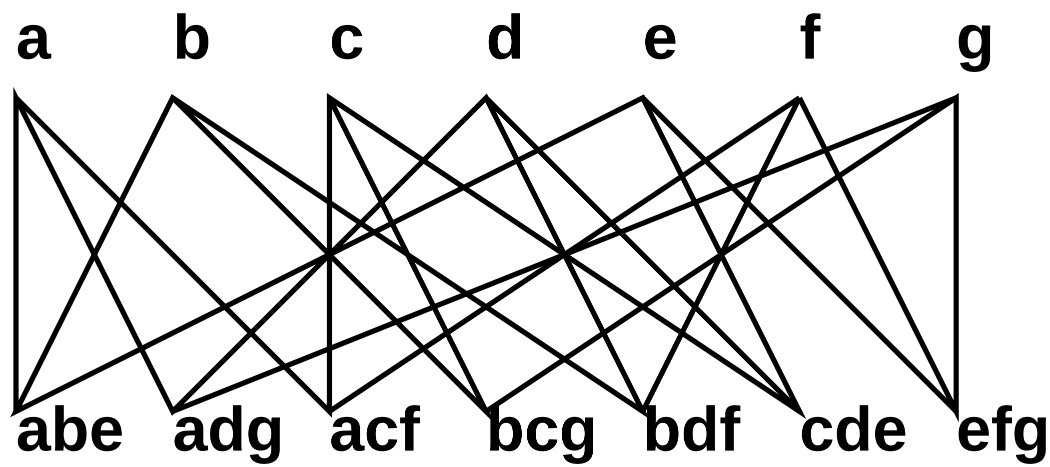
Designs



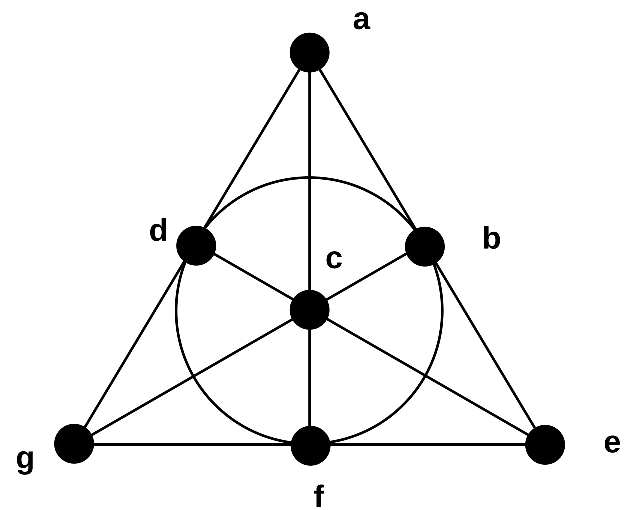
Heawood graph



Heawood graph



Fano plane



Designs over Finite Fields

- a set of v points
- a set of $k-$blocks
- $t-(v,k,\lambda)$ Design
  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs over Finite Fields

- ~~a set of v points~~

  linear $v-$space $GF(q)^v$
- a set of $k-$blocks

- $t-(v,k,\lambda)$ Design

  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs over Finite Fields

- ~~a set of v points~~

  linear $v-$space $GF(q)^v$

- ~~a set of $k-$blocks~~

  a set of $k-$spaces in $GF(q)^v$

- $t-(v,k,\lambda)$ Design

  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs over Finite Fields

- ~~a set of v points~~

  linear $v-$space $GF(q)^v$
- ~~a set of $k-$blocks~~

  a set of $k-$spaces in $GF(q)^v$
- ~~$t-(v,k,\lambda)$ Design~~

  ~~each $t-$set of points is in exactly λ blocks~~

  $t-(v,k,\lambda)$ $q-$Design

  each $t-$space of $GF(q)^v$ is in exactly λ of the $k-$spaces

UNIVERSITÄT BAYREUTH

Current State

known:

- Thomas (1987): first to study, $2-$designs
- Braun, Kerber, Laue (2005): first $3-$design

open problems:

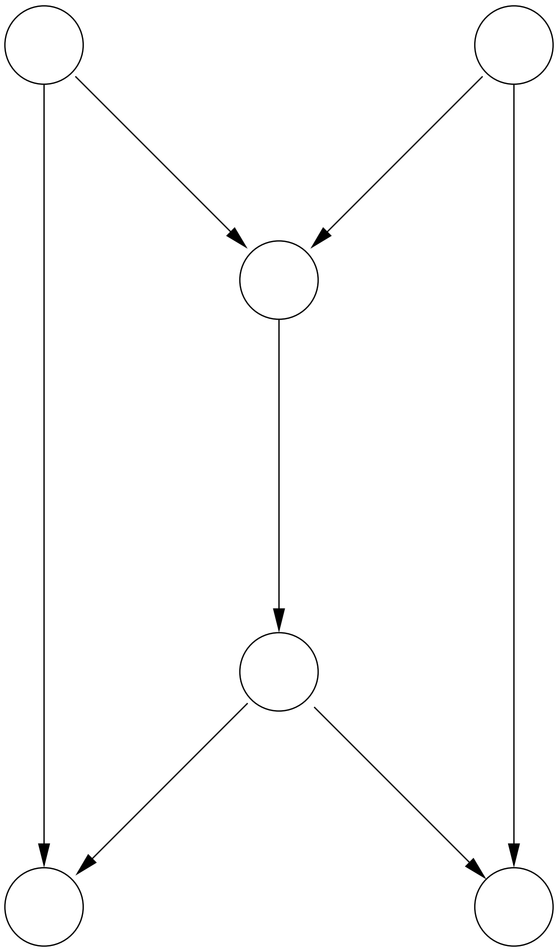
- $q-$analog of the Fano plane?
- Steiner systems ? ($\lambda = 1$)
- $t > 3$?

UNIVERSITÄT BAYREUTH

Network Codes

Network Codes

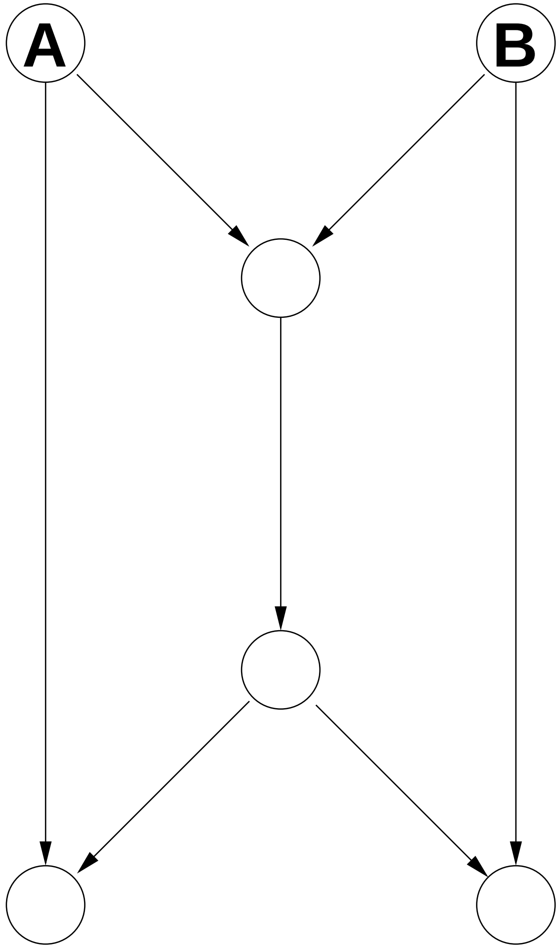
Sender



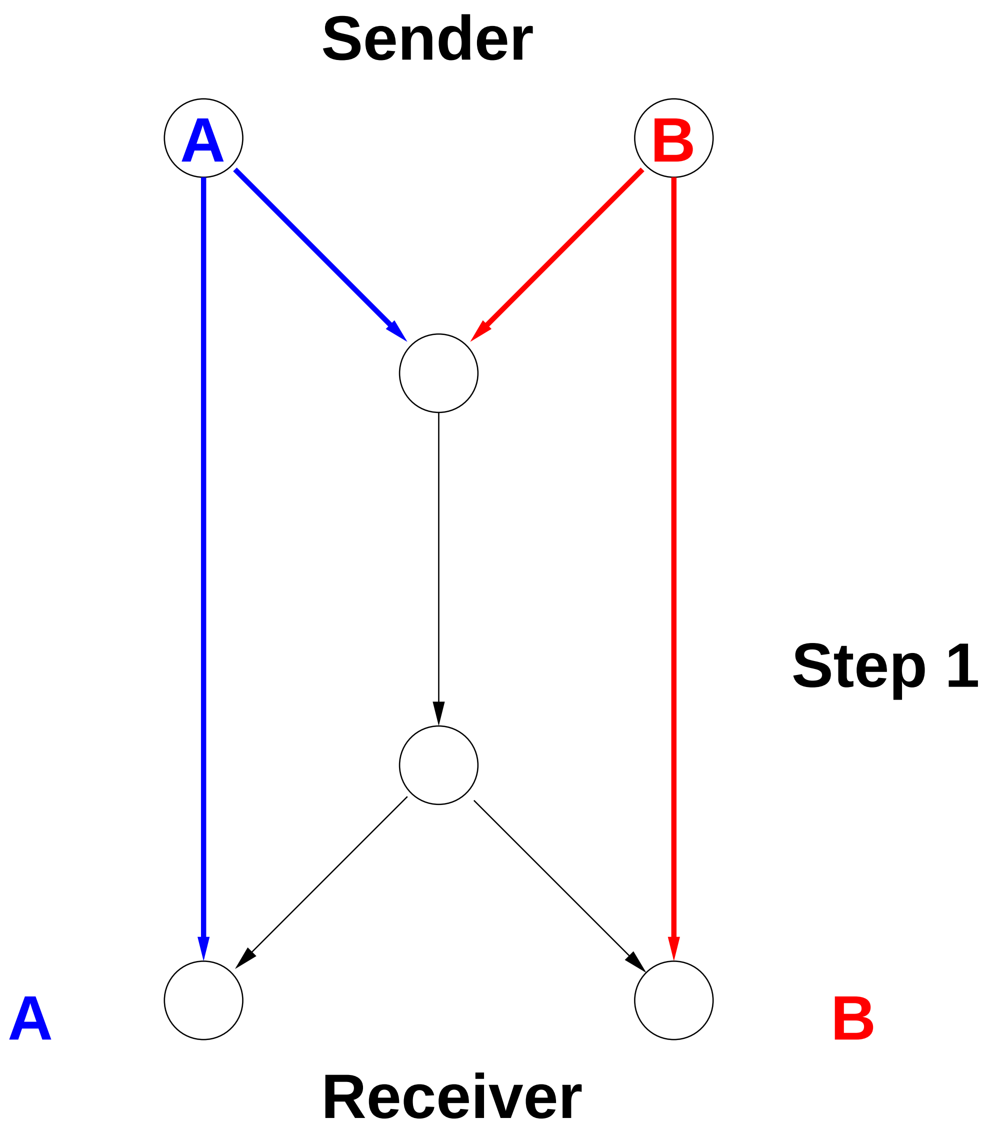
Receiver

Network Codes

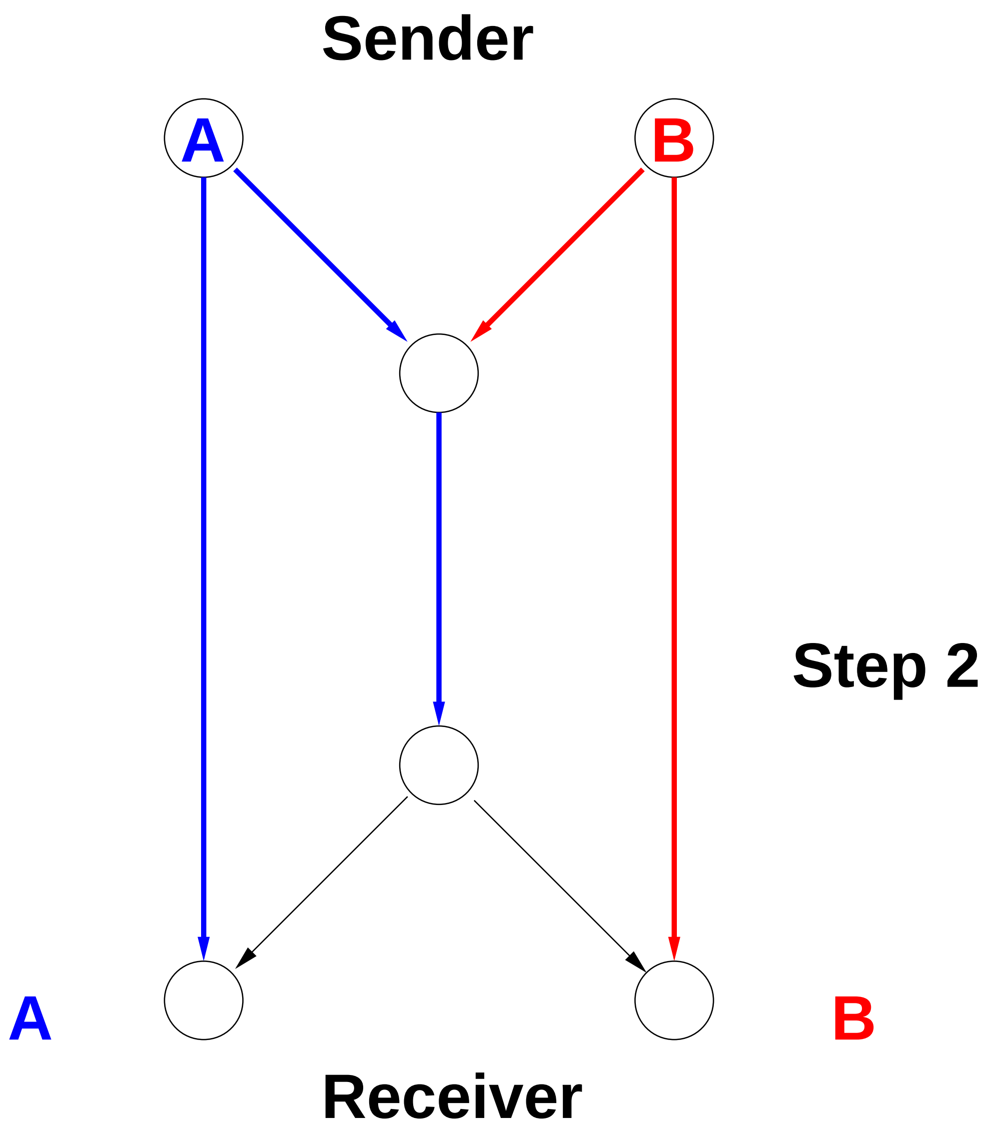
Sender



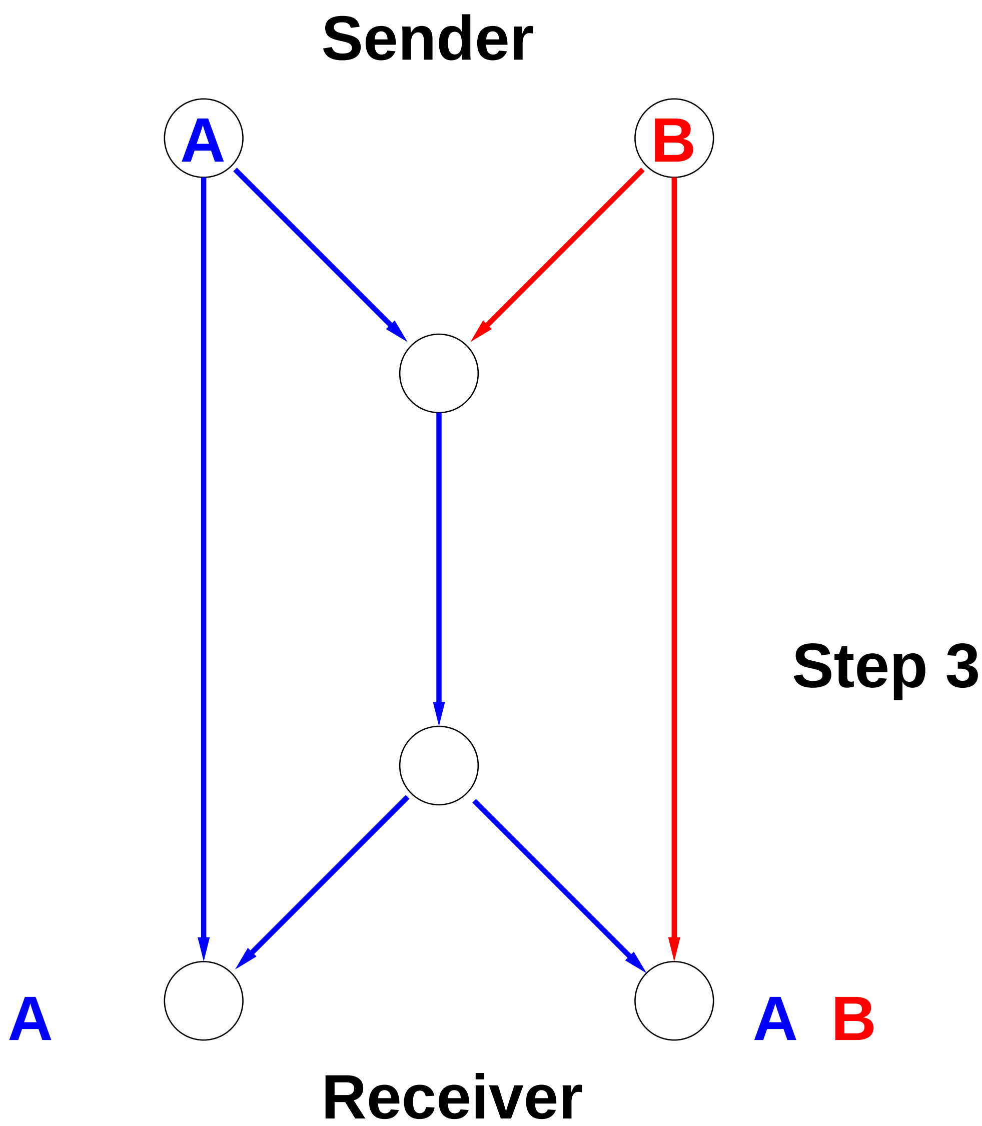
Receiver


Sender
A
B
Step 1
A
B
Receiver

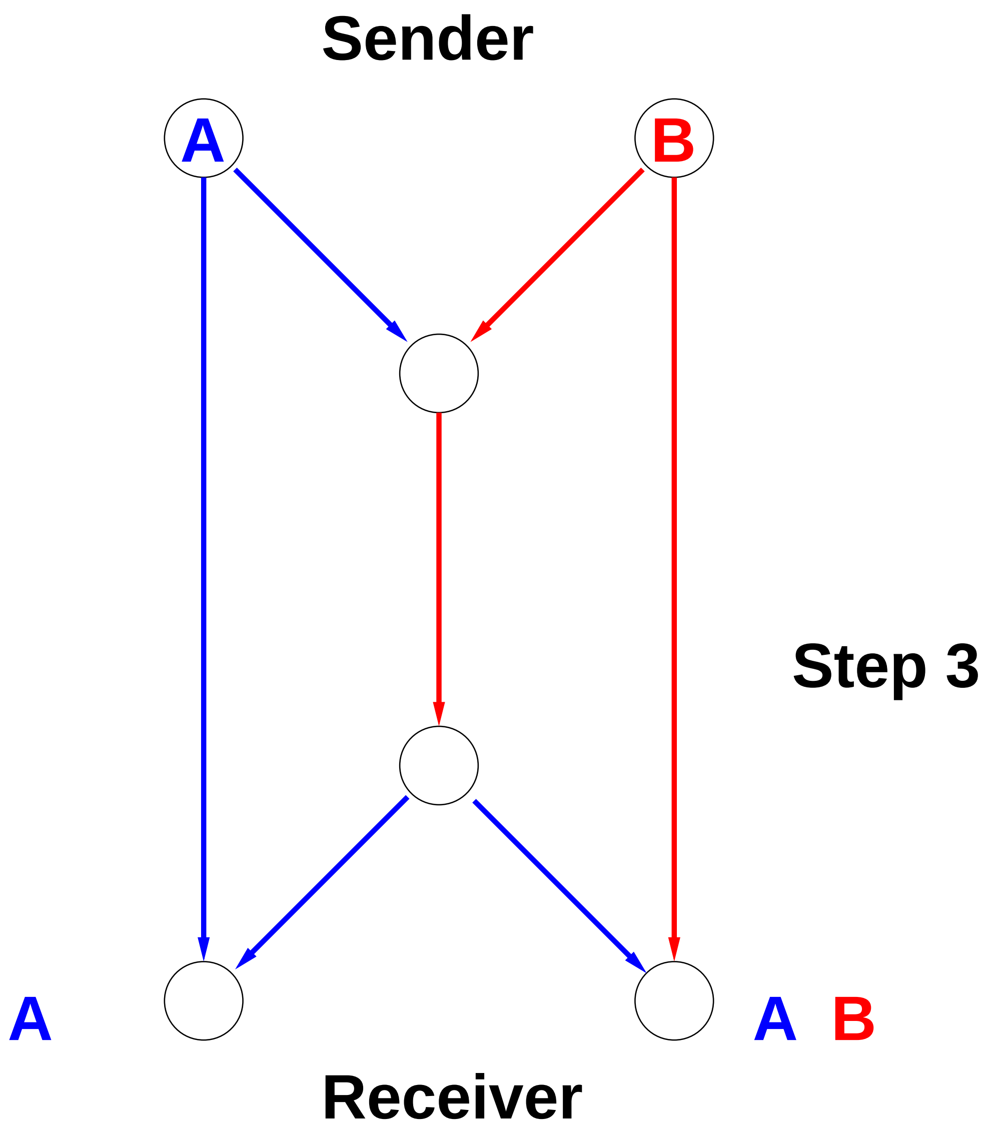
Network Codes



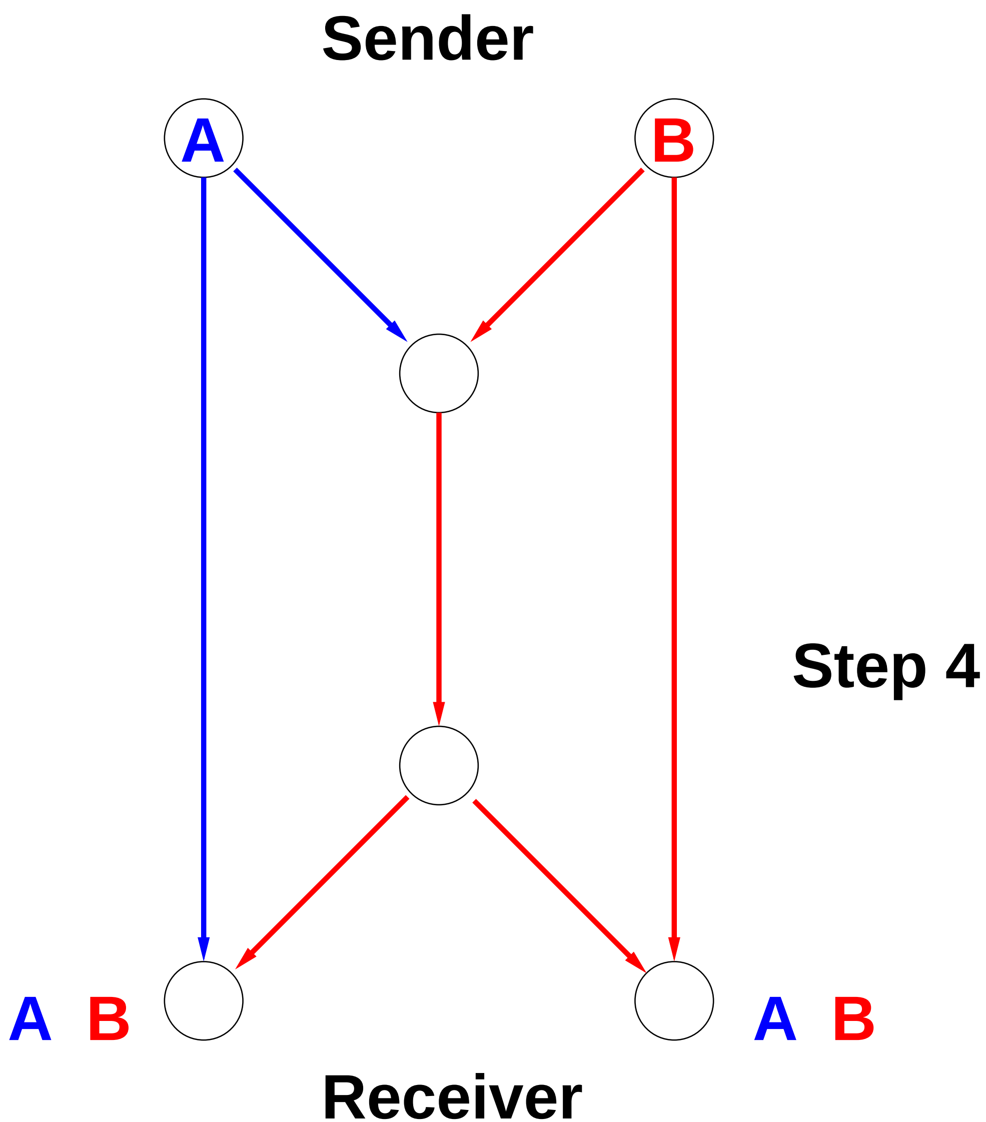
Network Codes



UNIVERSITÄT BAYREUTH

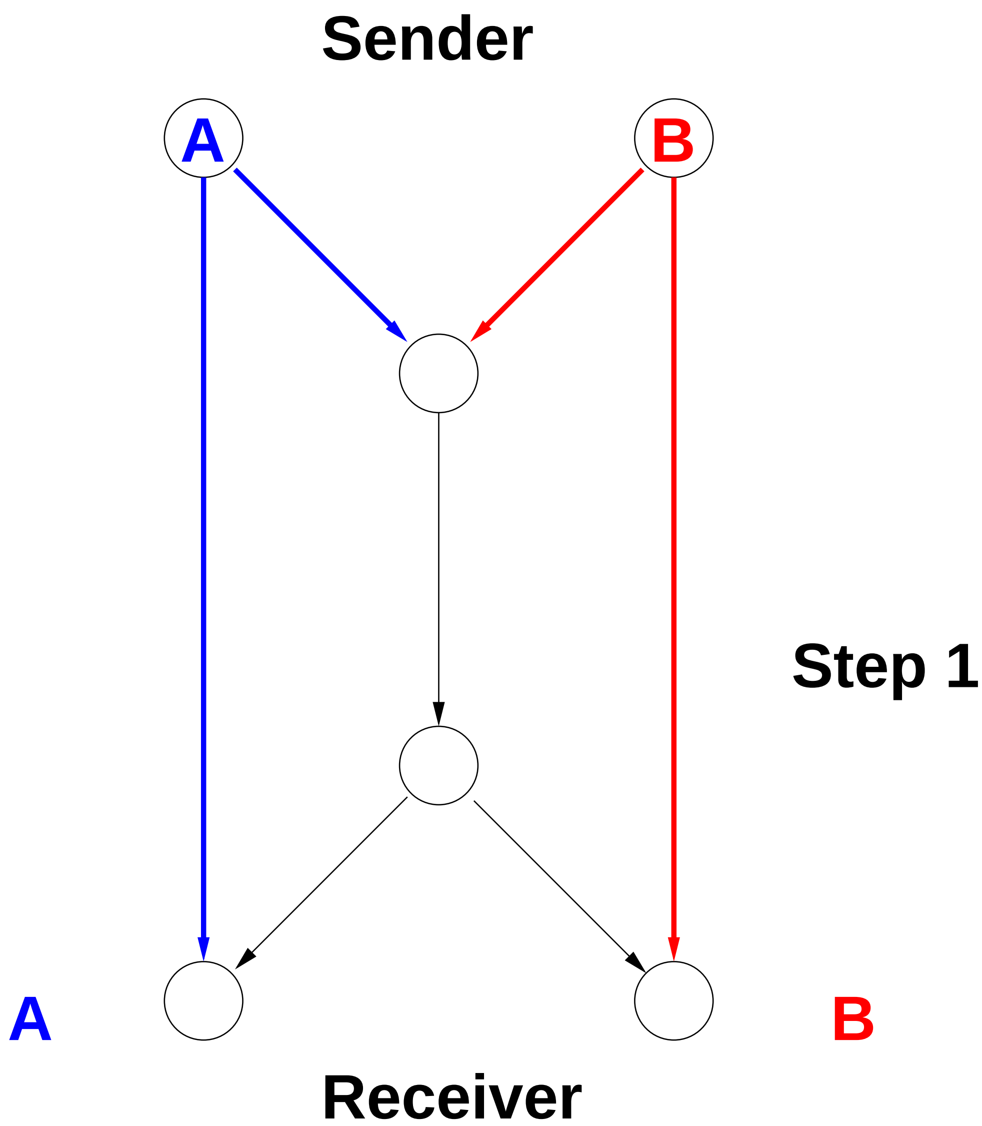

Sender
A
B
Step 3
A
A B
Receiver

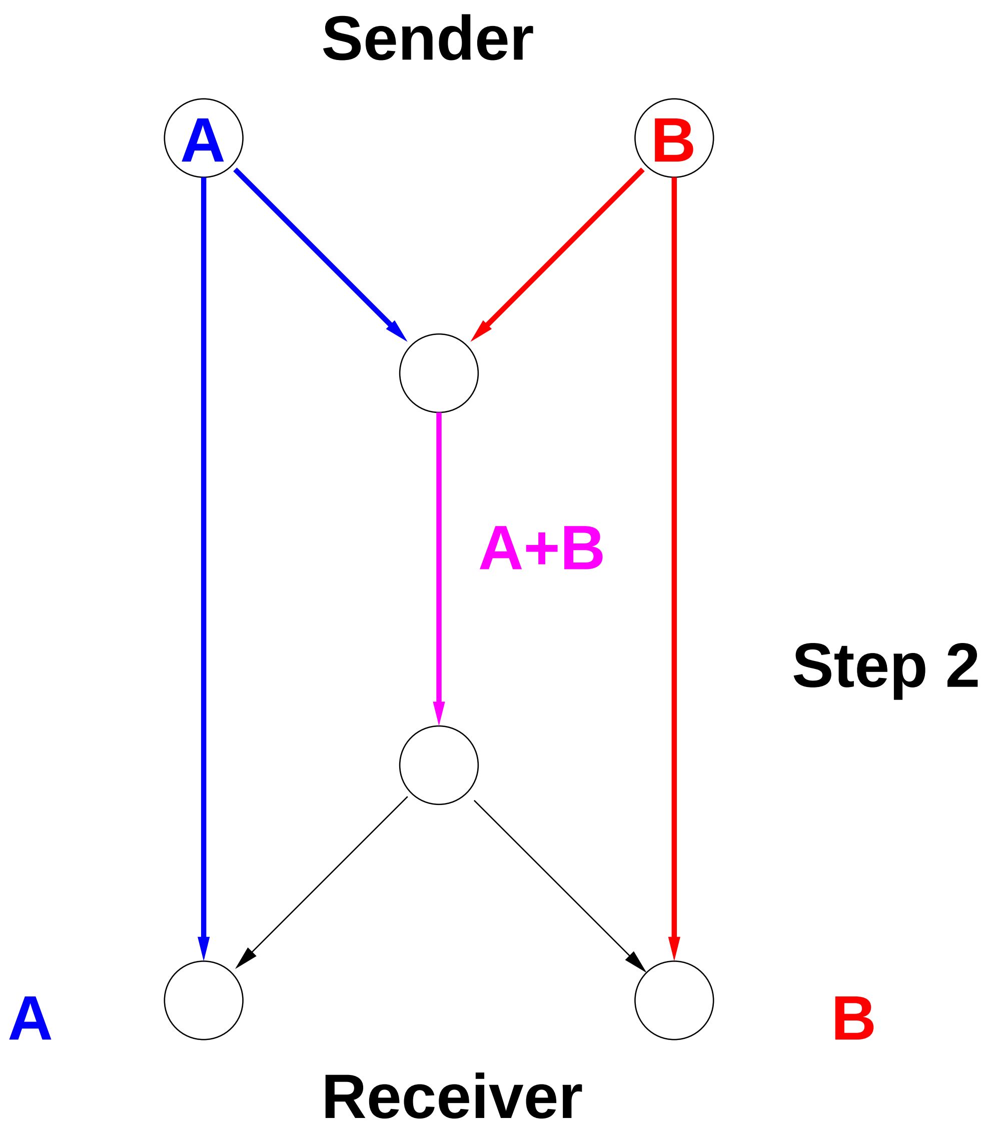
Network Codes

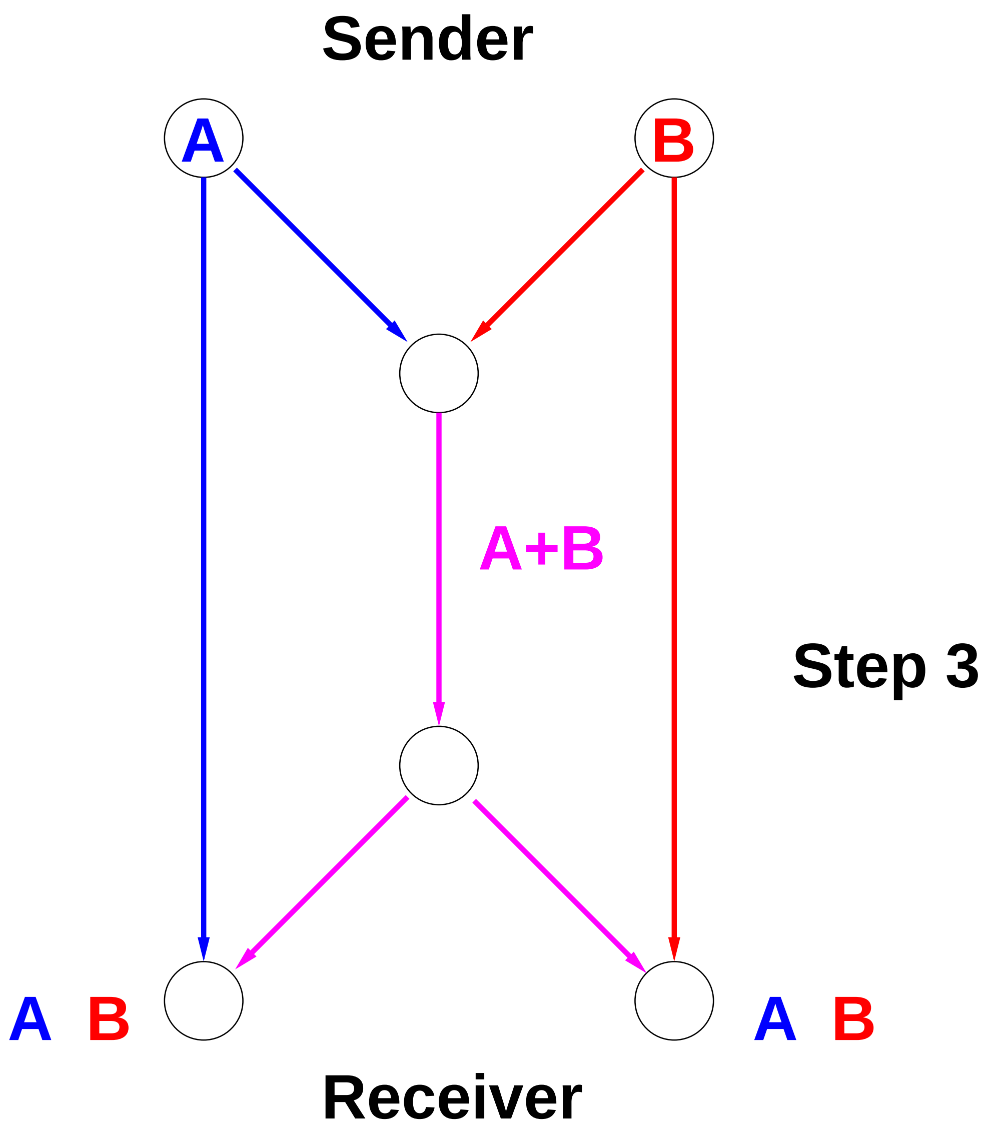


UNIVERSITÄT BAYREUTH

Network Codes




Sender
A
B
A+B
Step 2
A
B
Receiver


Sender
A
B
A+B
Step 3
A B
A B
Receiver

Network Codes

message:

- linear space

Network Codes

message:

- linear space

single node:

- receives vectors
- sends some linear combination of the incoming vectors

Error-Correcting Network Codes

codeword:

- linear subspace of $GF(q)^v$

Error-Correcting Network Codes

codeword:

- linear subspace of $GF(q)^v$

distance d:

- distance in the Hasse diagram of the linear lattice of all subspaces of $GF(q)^v$

Error-Correcting Network Codes

codeword:

- linear subspace of $GF(q)^v$

distance d:

- distance in the Hasse diagram of the linear lattice of all subspaces of $GF(q)^v$

U, W subspace of $GF(q)^v$:

$$d(U,W) = dim(U) + dim(W) - 2dim(U \cap W)$$

UNIVERSITÄT BAYREUTH

Error-Correcting Network Codes

fix minimum distance d:

$$\boxed{\text{Find a set of subspaces in } GF(q)^v \text{ such that the pairwise distance is at least } d}$$

Error-Correcting Network Codes

fix minimum distance d:

Find a set of subspaces in $GF(q)^v$ such that the pairwise distance is at least d

fix also dimension k of the subspaces:

Find a set of $k-$subspaces in $GF(q)^v$ such that the pairwise distance is at least $2d$

UNIVERSITÄT BAYREUTH

Error-Correcting Network Codes

fix minimum distance d:

$$\boxed{\text{Find a set of subspaces in } GF(q)^v \text{ such that the pairwise distance is at least } d}$$

fix also dimension k of the subspaces:

$$\boxed{\text{Find a set of } k\text{−subspaces in } GF(q)^v \text{ such that the pairwise distance is at least } 2d}$$

constant dimension codes $\approx$ constant weight codes

UNIVERSITÄT
BAYREUTH

Codes and Designs

Given a $t-(v,k,1)$ $q-$design we get a constant dimension code with minimum distance $2(k-(t-1))$ as the intersection of two codewords has dimension $\leq t-1$.

UNIVERSITÄT BAYREUTH

Codes and Designs

Given a $t-(v,k,1)$ $q-$design we get a constant dimension code with minimum distance $2(k-(t-1))$ as the intersection of two codewords has dimension $\leq t-1$.

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in exactly 1 $k-$subspace
= Steiner system

UNIVERSITÄT BAYREUTH

Given a $t-(v,k,1)$ $q-$design we get a constant dimension code with minimum distance $2(k-(t-1))$ as the intersection of two codewords has dimension $\leq t-1$.

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in exactly 1 $k-$subspace
= Steiner system

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in at most 1 $k-$subspace
= error-correcting network code

UNIVERSITÄT BAYREUTH

Current State

Define $A_q(v,k,d)$ as the maximal size (= number of codewords) of a constant dimension code with minimum distance d, dimension of codewords = k, and ambient space = $GF(q)^v$

UNIVERSITÄT BAYREUTH

Current State

Define $A_q(v,k,d)$ as the maximal size (= number of codewords) of a constant dimension code with minimum distance d, dimension of codewords = k, and ambient space = $GF(q)^v$

open problems:

- find lower and upper bounds for $A_q(v,k,d)$
- find constructions of 'good' codes
- special case $A_2(7,3,4)$ = Fano plane

UNIVERSITÄT BAYREUTH

Construction

Problem

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in at most 1 $k-$subspace
= error-correcting network code

Problem

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in at most 1 $k-$subspace
= error-correcting network code

D:= incidence matrix between $k-$spaces and $t-$spaces in $GF(q)^v$

$$D_{U,V} := \begin{cases} 1 & t\text{-space } U \text{ is subspace of } k-\text{space } W \\ 0 & \text{else} \end{cases}$$

UNIVERSITÄT BAYREUTH

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

- $x_1 + \ldots + x_s$ as large as possible

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

- $x_1 + \ldots + x_s$ as large as possible
- $Dx^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

- $x_1 + \ldots + x_s$ as large as possible
- $Dx^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

solution = network code with minimum distance $2(k - t + 1)$.

Automorphisms

Automorphism φ on $GF(q)^v$: $U < W \iff U^\varphi < W^\varphi$

G subgroup of $Aut(GF(q)^v)$

UNIVERSITÄT BAYREUTH

Automorphisms

Automorphism φ on $GF(q)^v$: $U < W \iff U^\varphi < W^\varphi$
G subgroup of $Aut(GF(q)^v)$

- shrink matrix D by: adding columns of elements in the same orbit of G on the $k-$spaces

$\Rightarrow$ rows of elements in the same orbit on the $t-$spaces are identical

UNIVERSITÄT BAYREUTH

Automorphisms

Automorphism φ on $GF(q)^v$: $U < W \iff U^\varphi < W^\varphi$
G subgroup of $Aut(GF(q)^v)$

- shrink matrix D by: adding columns of elements in the same orbit of G on the $k-$spaces

$\Rightarrow$ rows of elements in the same orbit on the $t-$spaces are identical

- D^G $:=$shrinked matrix

$\Rightarrow$number of columns = number of orbits on $k-$spaces
number of rows = number of orbits on $t-$spaces

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

- $b_1 x_1 + \ldots + b_m x_m$ as large as possible

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

- $b_1 x_1 + \ldots + b_m x_m$ as large as possible
- $D^G x^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

UNIVERSITÄT BAYREUTH

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

- $b_1 x_1 + \ldots + b_m x_m$ as large as possible
- $D^G x^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

solution = network code with prescribed automorphisms and minimum distance $2(k-t+1)$.

UNIVERSITÄT BAYREUTH

Results (binary)

v	k	number of codewords: new	old	d
6	3	77	71	4
7	3	304	294	4
8	3	1275	1164	4
9	3	5621	4657	4
10	3	21483	18631	4
11	3	79833	74531	4
12	3	315315	298139	4

Last Page

A. Kohnert, S. Kurz: *Construction of Large Constant Dimension Codes With a Prescribed Minimum Distance*, LNCS, 2008.

R. Kötter, F. Kschischang: *Coding for errors and erasures in random network coding*, IEEE Transactions on Information Theory, **54**, 3579–3590, 2008.

Thank you very much for your attention.

UNIVERSITÄT BAYREUTH