Constructive methods for the computation of Schubert polynomials

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Melbourne February 2008

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From the lecture of Paul Zinn-Justin:

Definition (Schubert polynomials)

The Schubert polynomial $X_\sigma(x_1, \dots, x_N; y_1, \dots, y_N)$ is defined as

$$X_\sigma := \mathrm{mdeg}_{\mathfrak{g}}^T\, S_\sigma$$

From the lecture of Alain Lascoux:

Definition 1 *Given $v \in \mathbb{N}^n$, the Schubert polynomial $Y_v(\mathrm{x})$, also denoted $X_\sigma(\mathrm{x})$ with $\sigma = \langle v \rangle$, is the only polynomial in $\mathfrak{Pol}_{|v|}(\mathrm{x}, \mathrm{y})$ such that*

$$Y_v(\mathbf{y}^{\langle u \rangle}) = 0\,,\ u \neq v,\ |u| \leq |v| \qquad (1)$$

$$Y_v(\mathbf{y}^{\langle v \rangle}) = \Cap(v) := \prod_{i<j,\, \sigma_i > \sigma_j} (y_{\sigma_i} - y_{\sigma_j}) \qquad (2)$$

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Overview

- Combinatorial description of Schubert polynomials $X_\pi(x)$
- Combinatorial description of double Schubert polynomials $X_\pi(x, y)$
- Properties of Schubert polynomials

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Combinatorial Description

- Permutation $\pi = [\pi_1, \pi_2, \ldots, \pi_n] \in S_n$

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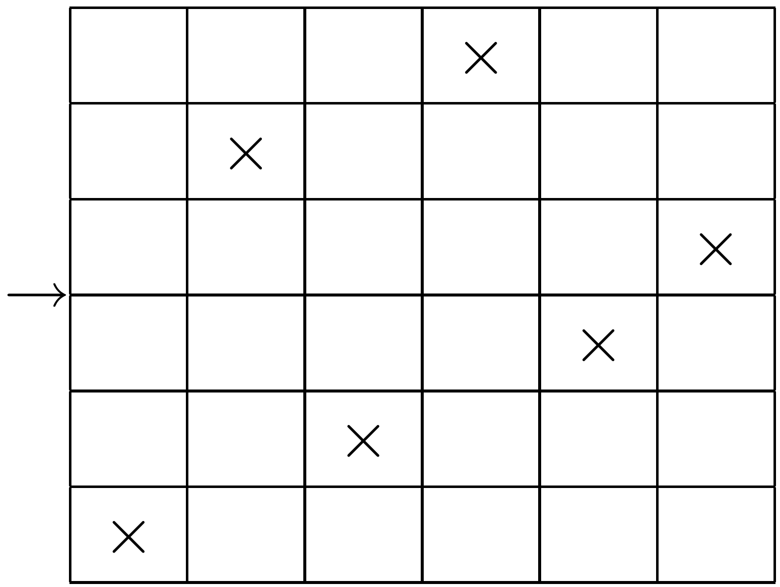
$[1, 3, 5, 6, 2, 4] \rightarrow$



Combinatorial Description

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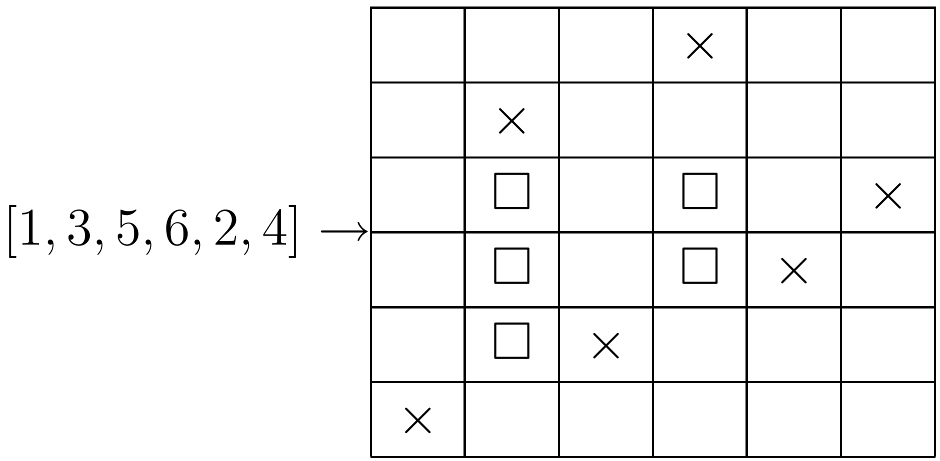
Each inversion one box in the Rothe diagram:



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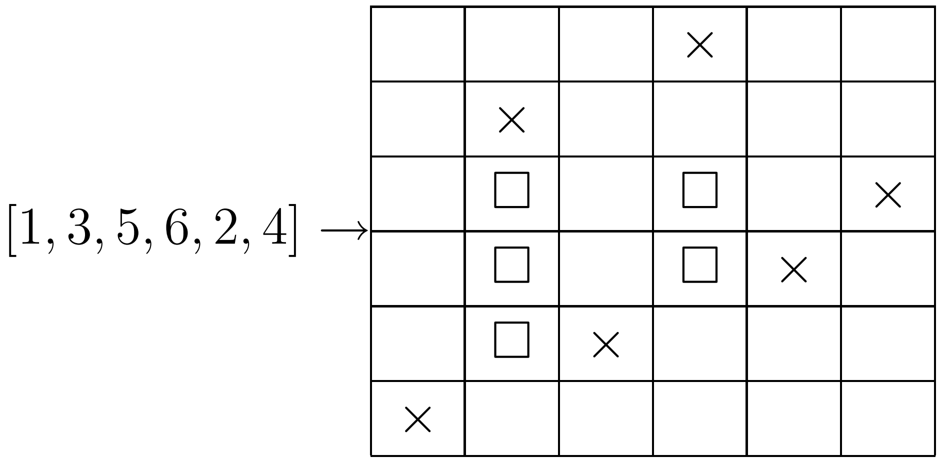

The *code* of permutation π is the vector $I = I_1, \ldots$ with I_i :=number of boxes in row i

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$[1, 3, 5, 6, 2, 4] \rightarrow$

			$\times$			0
	$\times$					0
	$\square$		$\square$		$\times$	2
	$\square$		$\square$	$\times$		2
	$\square$	$\times$				1
$\times$						0

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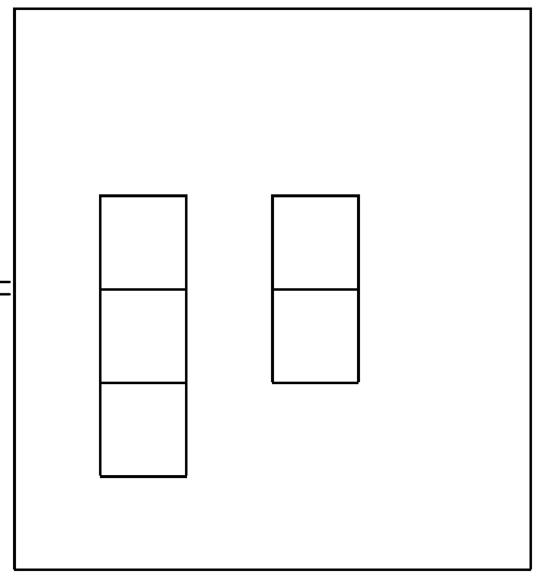
$$[1,3,5,6,2,4] \longrightarrow D_{135624} =$$



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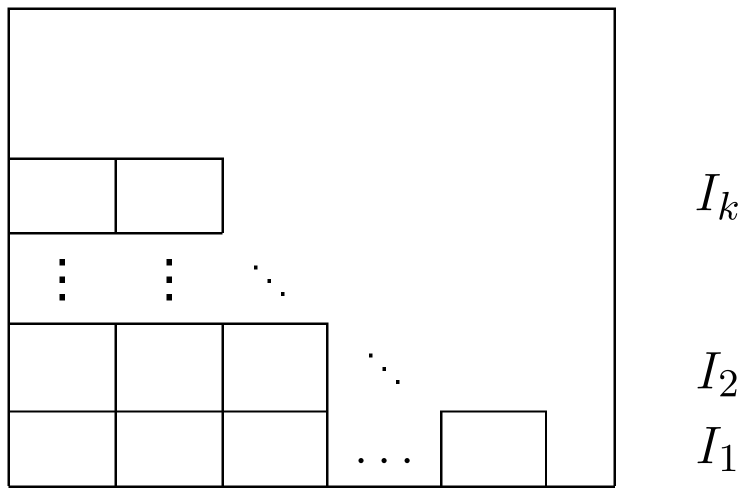

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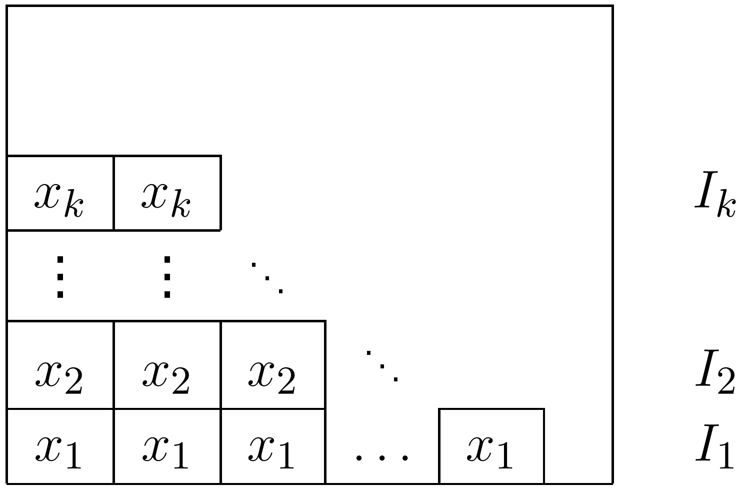

To get the Schubert polynomial $Y_I(x)$:
Each box in row i becomes a factor x_i , and then multiply the boxes.

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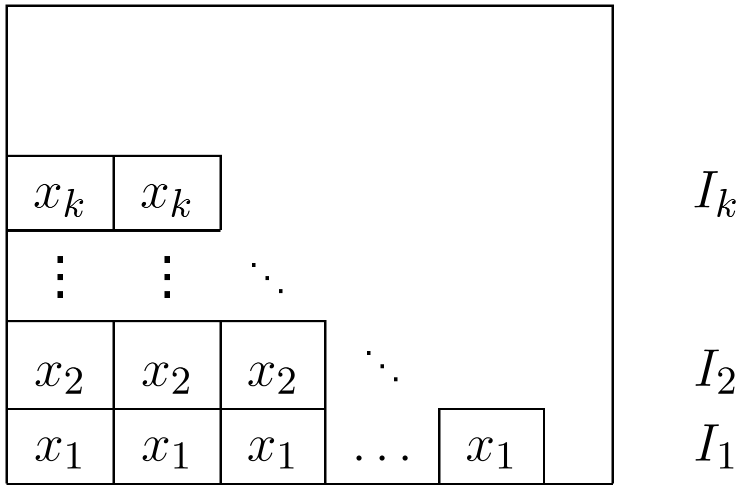

To get the Schubert polynomial $Y_I(x)$:
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$$Y_I(x)=x_1^{I_1}\ldots x_k^{I_k} = x^I$$

Combinatorial Description

code I $\longrightarrow$ diagram D $\longrightarrow$ $Y_I = ev(D)$

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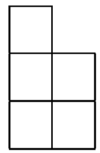
code I	$\rightarrow$	diagram D	$\rightarrow$	$Y_I = ev(D)$
221				$x_1^2 x_2^2 x_3$

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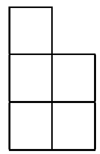

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more general

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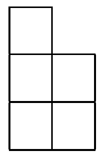

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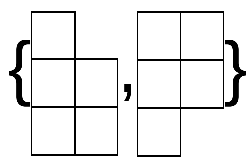

$x_1^2 x_2^2 x_3 + x_1 x_2^2 x_3^2$

Combinatorial Description

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Combinatorial Description

To get a set of diagrams, define moves r_i which modify a diagram D.

- $r_i(D) =$take the rightmost box in row i and exchange it with the first empty place below this box
- $S_1(D) :=$all diagrams which can be generated by a sequence of moves starting from D

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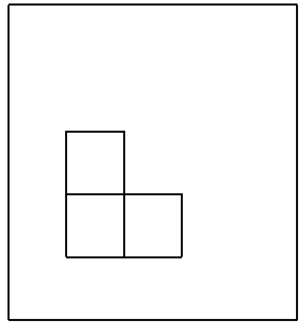
Combinatorial Description

$S_1(D_{14325})$



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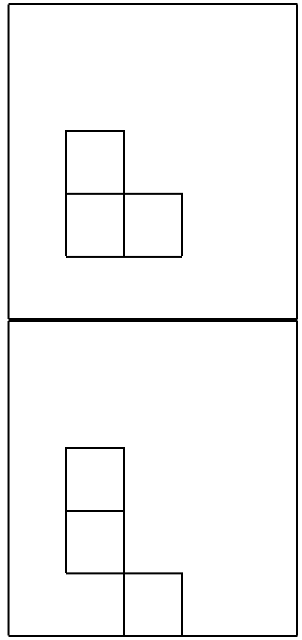

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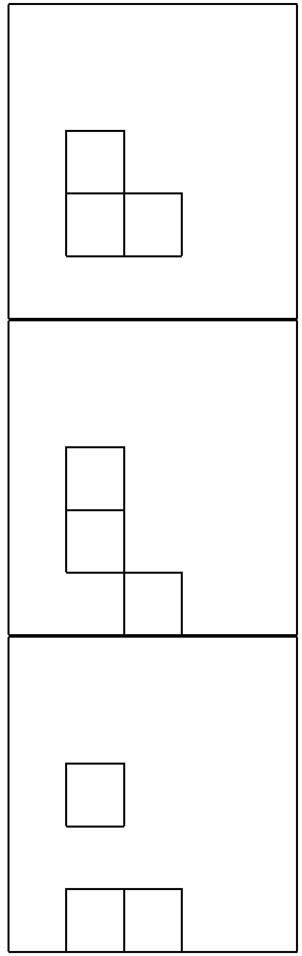

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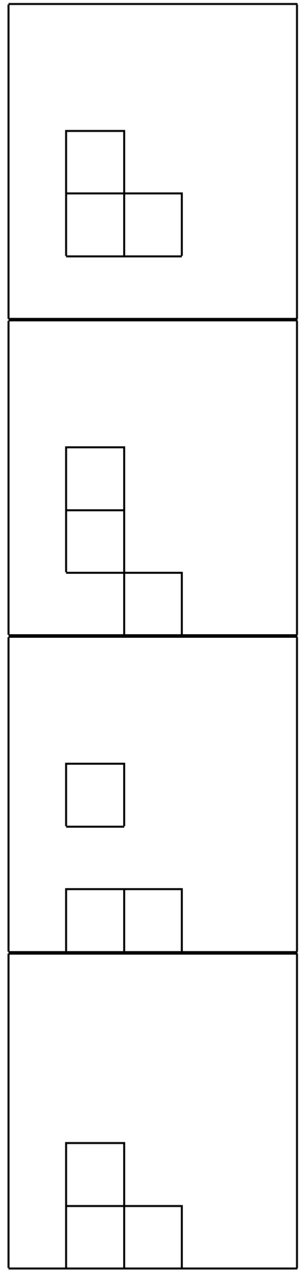

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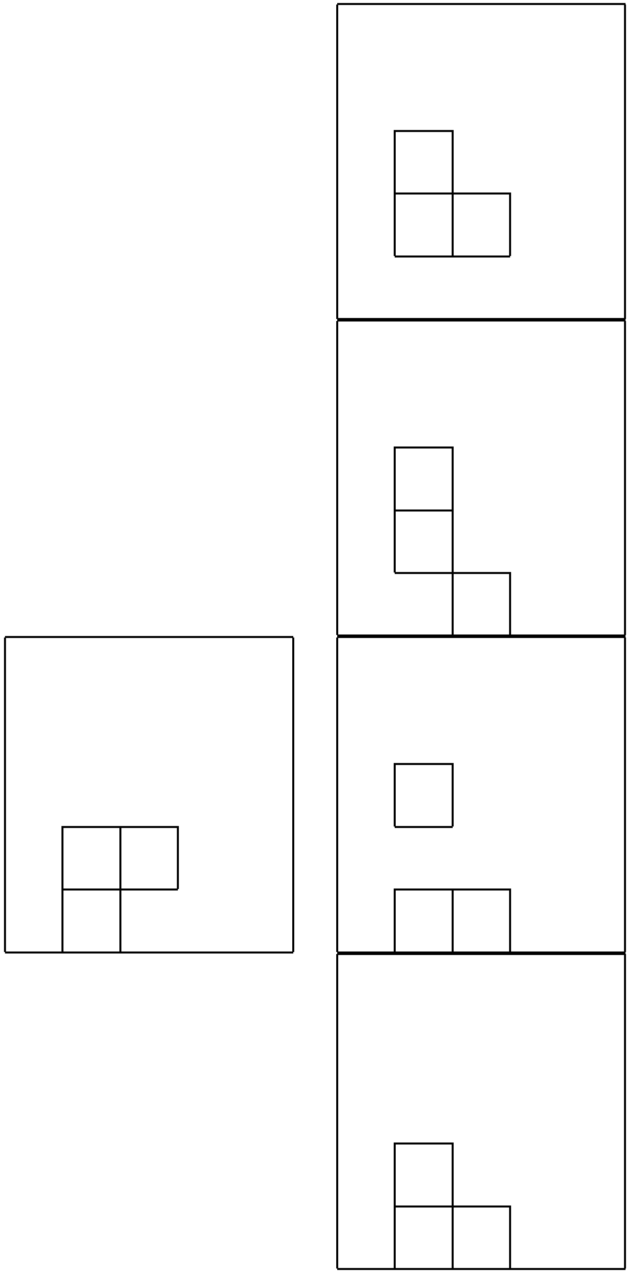

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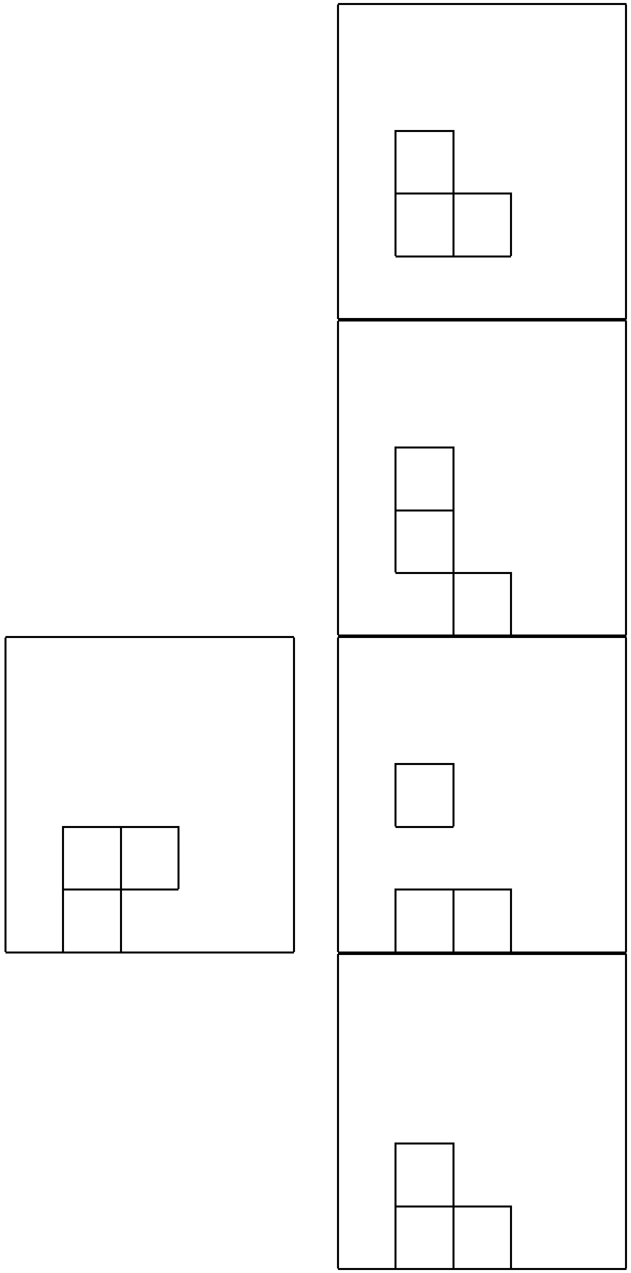

$\xrightarrow{ev} \quad x_2^2 x_3$

$\xrightarrow{ev} \quad x_1 x_2 x_3$

$\xrightarrow{ev} \quad x_1 x_2^2 + x_1^2 x_3$

$\xrightarrow{ev} \quad x_1^2 x_2$

Theorem (K., Winkel):

$$X_\pi(x) = \sum_{J \in S_1(D_\pi)} ev(J)$$

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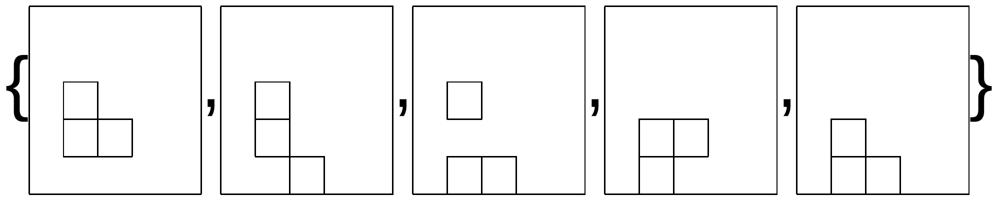
$$X_{1432}(x) = Y_{0210}(x) \to$$



$$\to x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2$$

Double Schubert Polynomial $X_\pi(x, y)$, $Y_I(x, y)$

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- modify start diagram
- modify moves
- modify evaluation

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The start diagram for double Schubert polynomials

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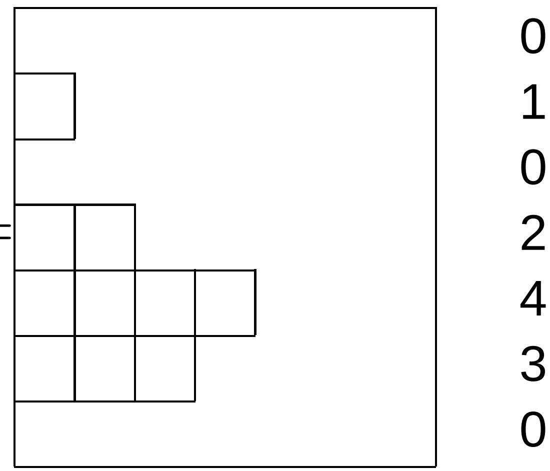
- $0342010 \to T_I =$ 

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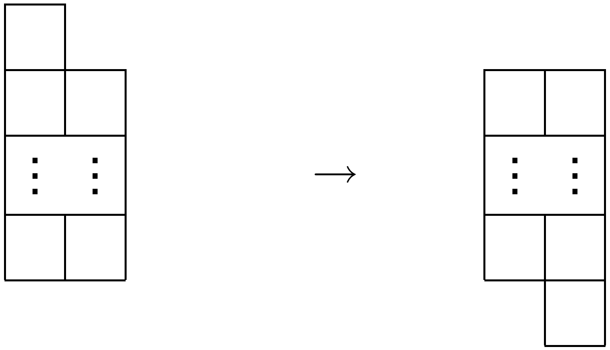

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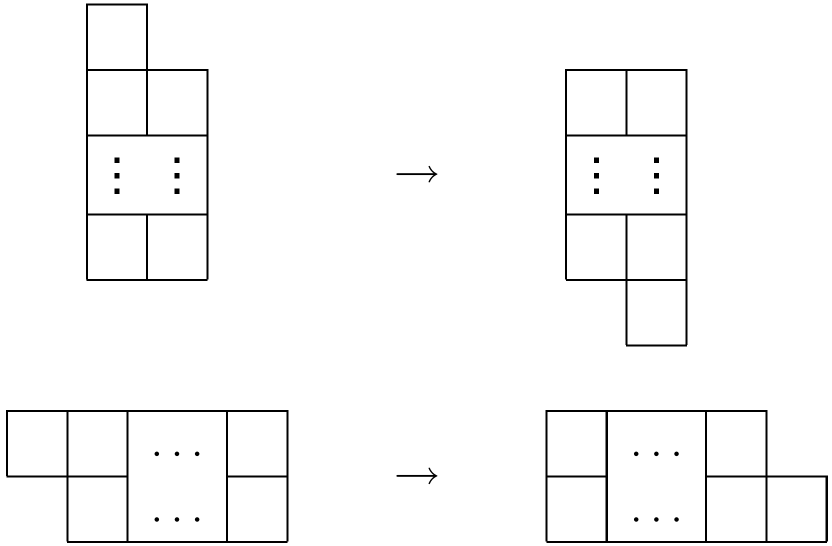

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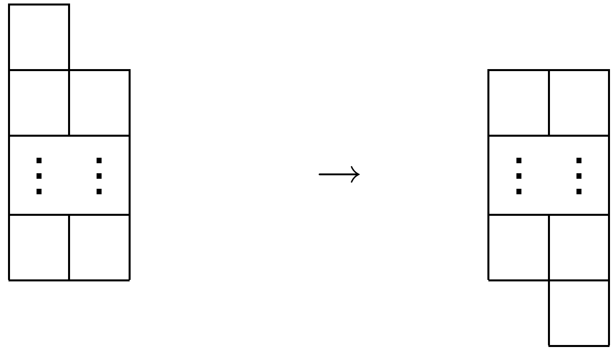

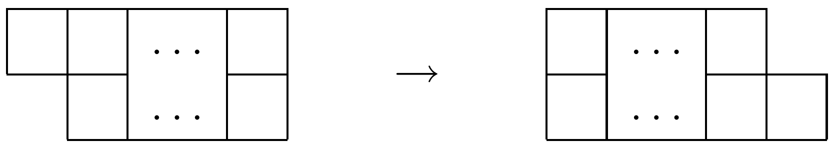

- the pairs of boxes below or to the right may be missing, then both moves coincide.

Combinatorial Description

The evaluation ev_2 used for double Schubert polynomials

- for each box at row i and column j in a diagram D take the factor $(x_i - y_j)$.
  $ev_2(D)$ is the product of all factors.

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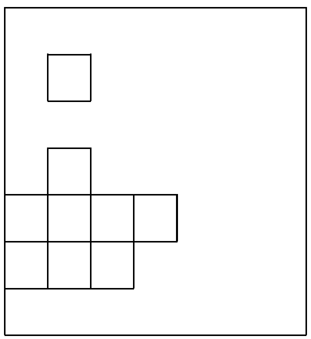
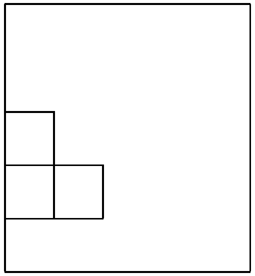
- 
  $$\xrightarrow{ev_2} \begin{array}{c} (x_6 - y_2) \\ (x_4 - y_2) \\ (x_3 - y_1)(x_3 - y_2)(x_3 - y_3)(x_3 - y_4) \\ (x_2 - y_1)(x_2 - y_2)(x_2 - y_3) \end{array}$$

Combinatorial Description

$S(D_{14325})$



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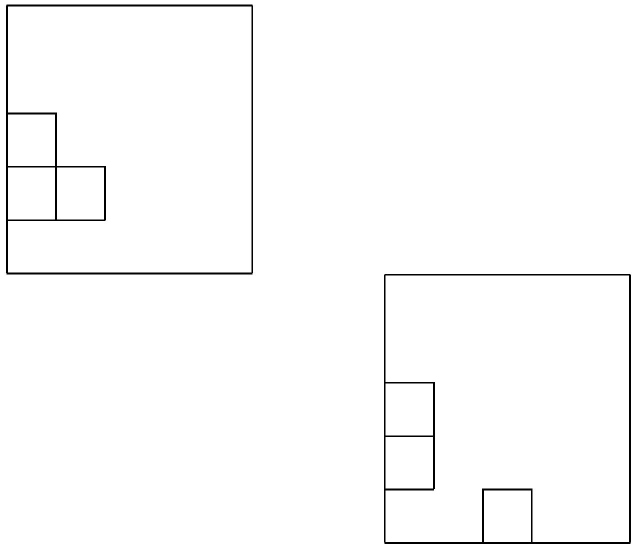

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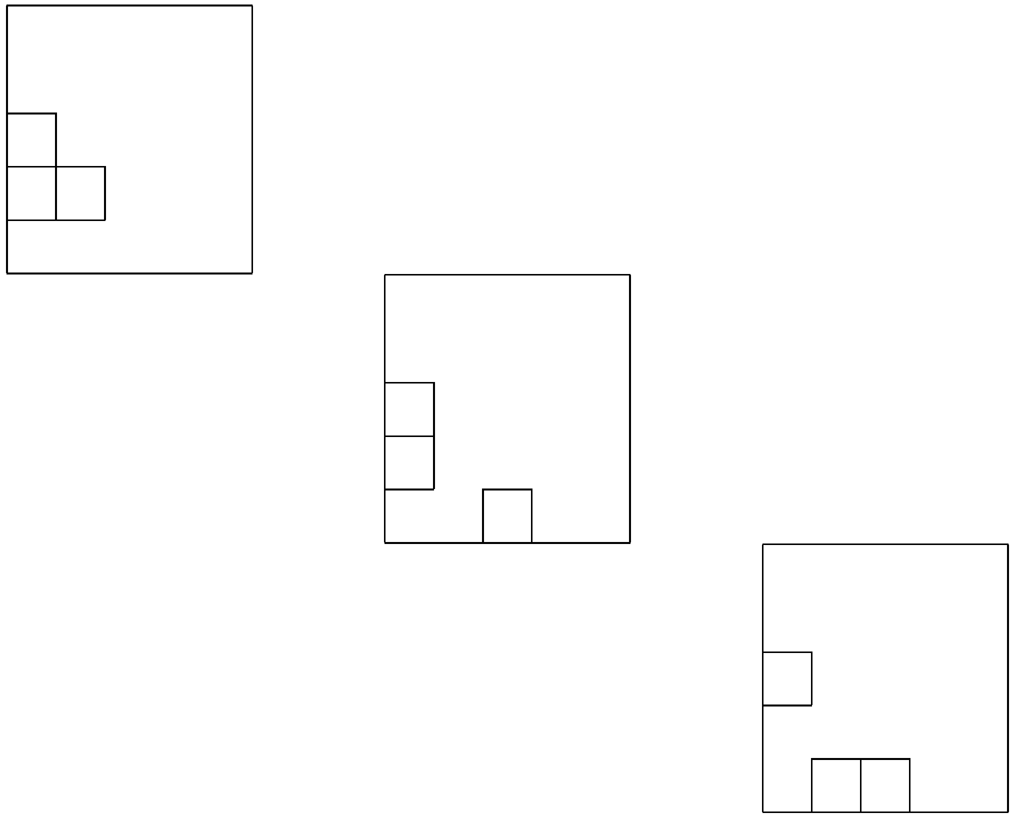

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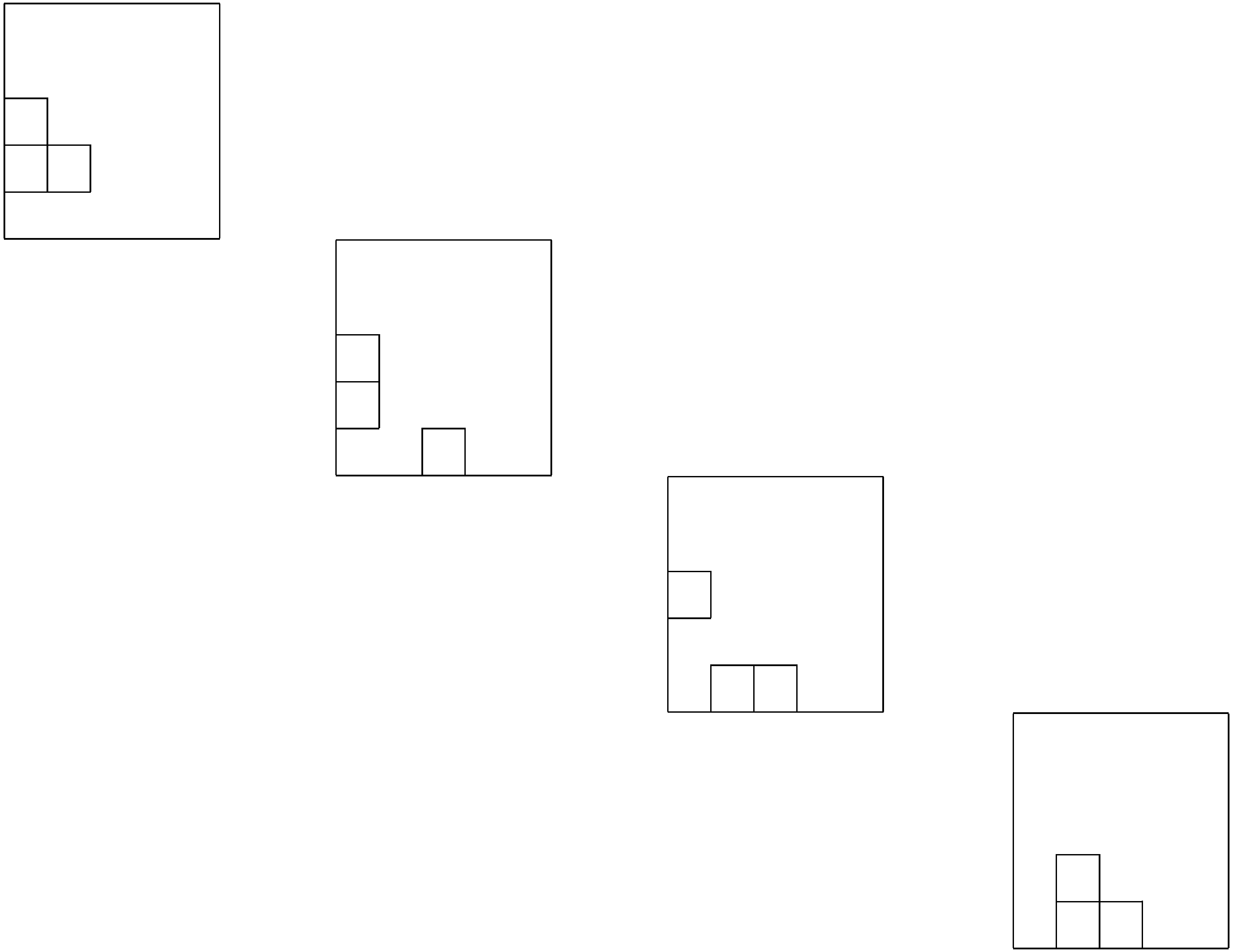

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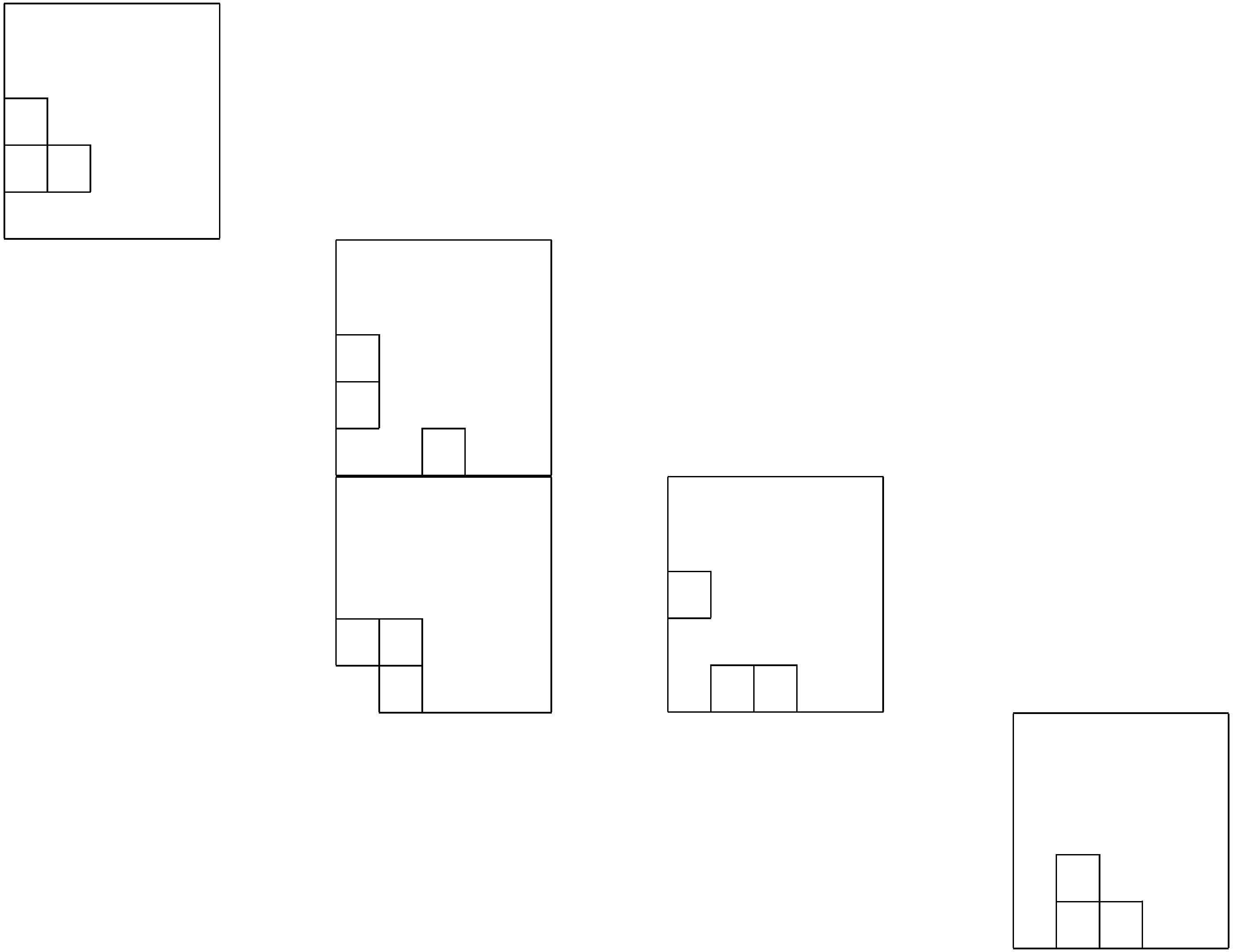

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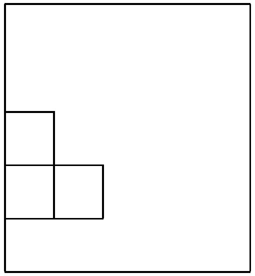
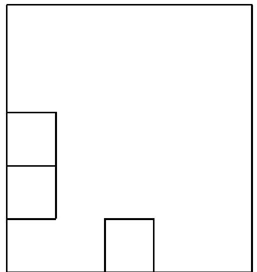


$\overset{ev_2}{\longrightarrow}$ $(x_3 - y_1)$

$(x_2 - y_1)(x_2 - y_2)$



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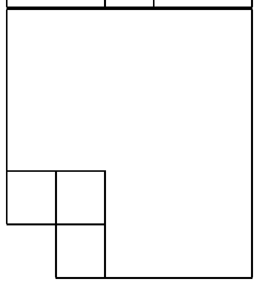
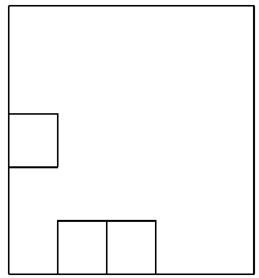
$(x_1 - y_2)$





$\overset{ev_2}{\longrightarrow}$ $(x_3 - y_1)$

$(x_1 - y_2)$ $(x_1 - y_3)$

$(x_2 - y_2)$ $\overset{ev_2}{\longleftarrow}$

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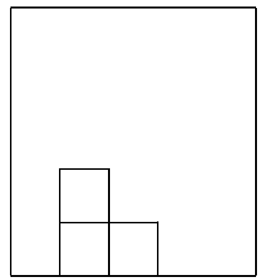

$$Y_{0210}(x,y) = (x_3 - y_1)(x_2 - y_1)(x_2 - y_2) + (x_3 - y_1)(x_2 - y_1)(x_1 - y_3) + \ldots$$

Combinatorial Description

Denote by $S_2(D)$ all diagrams generated from diagram D by a sequence of moves defined for double Schubert polynomials.

Theorem (Bergeron, Billey, K.):

$$Y_I(x,y) = \sum_{J \in S_2(T_I)} ev_2(J)$$

Factorization property:
$I = (I_1, \ldots, I_n)$ code of $\pi \in S_n$, $J = (J_1, \ldots, J_m)$ code of $\rho \in S_m$ then:

$$Y_{(I_1 \ldots I_n J_1 \ldots J_m)} = Y_I \cdot Y_{(\underbrace{0 \ldots 0}_{n} J_1 \ldots J_m)}$$

Properties

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$$D_{(\pi,\rho)} = \begin{array}{|c|c|} \hline \emptyset & D_\rho \\ \hline D_\pi & \emptyset \\ \hline \end{array}$$

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Schubert polynomial generalize Schur polynomials: Let $\lambda = \lambda_1 \geq \ldots \geq \lambda_k$ be a partition, then the Schur polynomial $S_\lambda(x_1, \ldots, x_n)$ ($n \geq k$) is equal to the Schubert polynomial $Y_{\underbrace{0\ldots0}_{n-k}\lambda_k\ldots\lambda_1\underbrace{0\ldots0}_{\lambda_1}}(x_1, \ldots, x_n)$.

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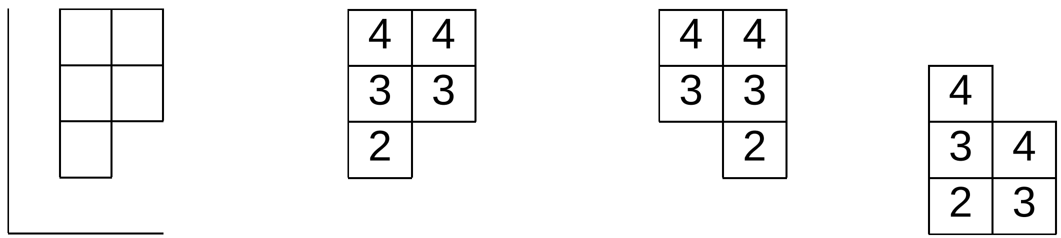
Schubert Schur

$D_\pi =$



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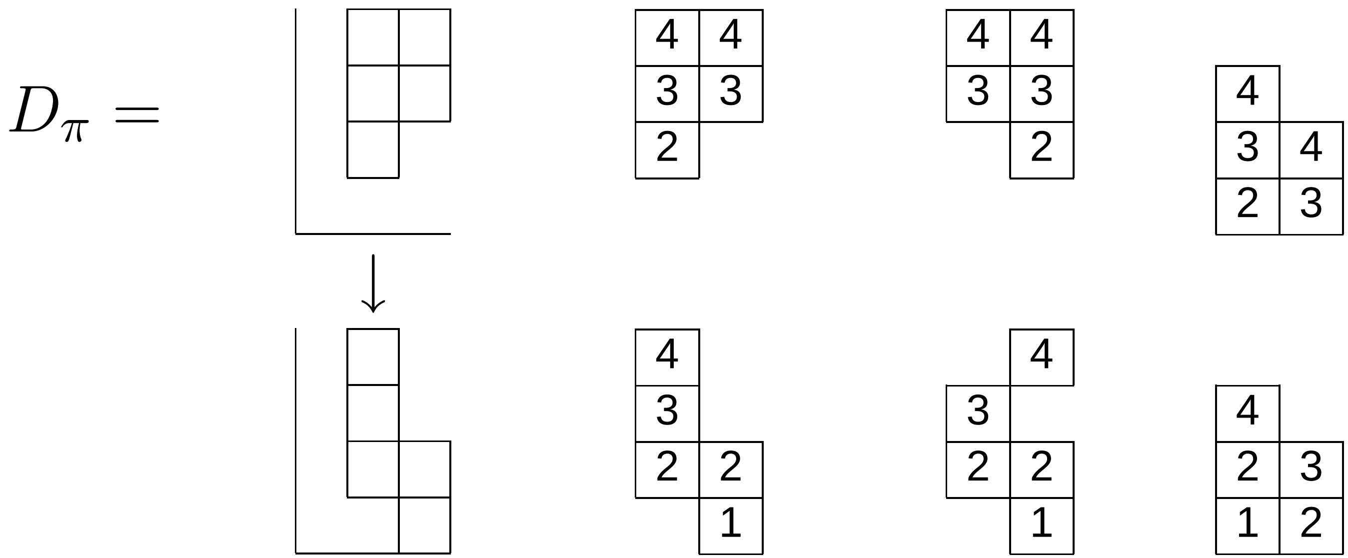
Schubert | Schur



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public domain package to compute with Schubert polynomials

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