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# Construction of Codes for Network Coding

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***Abstract*—Based on ideas of Kötter and Kschischang [6] we use constant dimension subspaces as codewords in a network. We show a connection to the theory of q-analogues of a combinatorial designs, which has been studied in [1] as a purely combinatorial object. For the construction of network codes we successfully modified methods (construction with prescribed automorphisms) originally developed for the q-analogues of a combinatorial designs. We then give a special case of that method which allows the construction of network codes with a very large ambient space and we also show how to decode such codes with a very small number of operations.**

network coding, q-analogue of Steiner systems, subspace codes, constant dimension subspace codes, decoding, Singer cycle

## I. INTRODUCTION

### A. Subspace Codes

In [6] R. Kötter and F. R. Kschischang developed the theory of subspace codes for applications in network coding. We will modify their presentation in some way. We denote by $L(GF(q)^v)$ the lattice of all subspaces of the *ambient* space, which is a vector space of dimension $v$ over the finite field with $q$ elements. The partial order of $L(GF(q)^v)$ is given by inclusion. A *subspace code* $C$ then is a subset of $L(GF(q)^v)$. A *constant dimension code* is the special case where all subspaces in $C$ are of the same dimension. To study error correcting codes we have to define some distance between codewords (in this case codewords are subspaces). The most natural one is the graph theoretic distance in the *Hasse diagram* (vertices are the elements of $L(GF(q)^v)$ and two subspaces are connected by an edge if they are direct neighbors in the partial order) of the lattice $L(GF(q)^v)$. An equivalent definition without using the underlying graph is as follows:

The *subspace distance* between two spaces $V$ and $W$ in $L(GF(q)^v)$ is defined as

$$d_S(V,W) := \dim(V+W) - \dim(V \cap W)$$

which is equal to

$$\dim(V) + \dim(W) - 2\dim(V \cap W).$$

This defines a metric on $L(GF(q)^v)$. Like in classical coding theory we define the *minimum (subspace) distance* of a subspace code $C$.

$$D_S(C) := \min\{d_S(V,W) : V, W \in C \text{ and } V \neq W\}.$$

We can now define the optimal (subspace) code problem:

> (P) For fixed parameters $q, v, d$ we want to find the maximal number $m$ of subspaces $V_1, \ldots, V_m$ in $L(GF(q)^v)$ such that the corresponding subspace code $C = \{V_1, \ldots, V_m\}$ has at least minimum distance $d$.

This is a specific instance of a packing problem in a graph. In classical coding theory the underlying graph is the Hamming graph. For network codes it is the Hasse diagram of the linear lattice. In the case of a binary code the Hamming Graph is isomorphic to the Hasse diagram of the powerset lattice. Using this connection we can look at the optimal (subspace) code problem as the $q-$*analogue* of the classical optimal code problem in the Hamming graph. To get the $q-$analogue we have to substitute a subset (= $0/1$ sequence of length $v$) of size $k$ by a $k-$dimensional subspace of $GF(q)^v$. The number of $k-$dimensional subspaces of $GF(q)^v$ is denoted by the Gaussian coefficient $\begin{bmatrix} v \\ k \end{bmatrix}_q$.

### B. $q-$Analogues of Designs

A $t-(v,k,\lambda)$ *design* is a set $C$ of $k-$element subsets (called blocks) of the set $\{1,\ldots,v\}$ such that each $t-$element subset of $\{1,\ldots,v\}$ appears in exactly $\lambda$ blocks. The special case of $\lambda = 1$ is called a *Steiner system*. Like in the subsection above we now define the $q-$analogue of a $t$-design. A $t-(v,k,\lambda)$ design over the finite field $GF(q)$ is a multiset $C$ of $k-$dimensional subspaces (called $q$-blocks) of the $v$-dimensional vector space $GF(q)^v$ such that each $t-$dimensional subspace of $GF(q)^v$ is a subspace of exactly $\lambda$ $q-$blocks. The connection with the constant dimension codes is given by the following observation in the case of a Steiner system: Given a $q-$analogue of a $t-(v,k,1)$ design $C$ we get a constant dimension code of minimum distance $2(k-t+1)$, since each $t$-dimensional space is contained in exactly one $k$-dimensional subspace the intersection between two spaces from $C$ is at most $(t-1)-$dimensional. Therefore the minimum distance of $C$ is at least $2(k-t+1)$. On the other hand given any $(t-1)-$dimensional subspace $V$ we can find two $t-$dimensional spaces $U, W$ with intersection $V$ and then two unique $q-$blocks containing $U$ and $W$. The minimum distance between these $q-$blocks is $2(k-t+1)$. $q$-analogues of designs were introduced by Thomas in 1987 [10]. Later they were studied in a paper by Braun et al. [1] where the authors constructed the first non-trivial $q-$analogue of a 3-design. We will describe a method in the second section which is based on their paper and which we use to construct constant dimension codes. First results found using that method were already in [7].

### *C. Encoding/Decoding*

In [6] the authors introduced the operator channel as a model to study subspace codes. The input and output alphabet of the channel is the lattice $L(GF(q)^v)$. The transmission of an information coded by a space $U$ works as follows: The transmitter inserts into the network vectors from $U$. During transmission the internal nodes of the network receive several such vectors and forward an arbitrary linear combination of the received vectors. The receiver collects the incoming vectors and tries to rebuild $U$ as the space generated by the incoming vectors. There are two possible problems at the receiver. There can be *erasures*, which means that some vectors are missing and the generated space is a subspace of $U$. Or there can be *errors*, which means that the receiver got vectors not from $U$. In the third section we will give a decoding algorithm for minimum distance decoding which allows error and erasure correction for a special class of constant dimension codes. This new method uses the symmetries of this special class of network codes, which were also studied by Etzion and Vardy [4], who called these codes cyclic.

## II. Construction of Constant Dimension Subspace Codes

In the following we work only with constant dimension codes built by a collection of $k-$dimensional subspaces of $GF(q)^v$ of minimum distance at least $2(k-t+1)$ as described in section I-B. In [7] we gave a general method using a prescribed group $G$ of automorphisms of a putative constant dimension code $C$ to give an equivalence between the existence of such a code and a solution of a Diophantine system of inequalities. Therefore the construction of such a code $C$ boils down to finding a $(0/1)-$solution $x$ of a Diophantine system of inequalities of the form:

$$M^G x \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

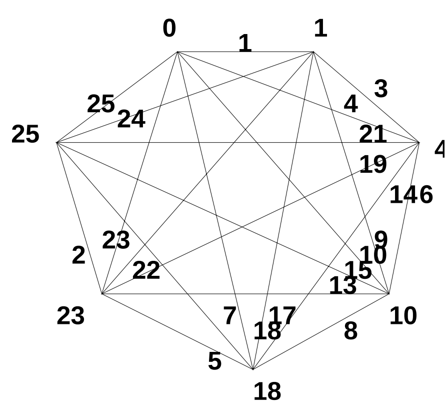
The number of rows in $M^G$ is the number of orbits of $G$ on the $t-$dimensional subspaces of $GF(q)^v$ and the number of columns is the number of orbits of $G$ on the $k-$dimensional subspaces of $GF(q)^v$. In [7] we described the general method for the construction using arbitrary automorphisms, and a further variant which is useful in the case of $G$ equal to the group generated by a Singer cycle $S$. Orbits of the Singer cycle has been studied for several applications [3]. The nice property of the Singer cycle is that $G = \langle S \rangle$ acts transitively on the one-dimensional subspaces of $GF(q)^v$. So we can label any one-dimensional subspace $W$ by the unique exponent $i$ between $0$ and $l := \begin{bmatrix} v \\ 1 \end{bmatrix}_q - 1$ with the property that $W = g^i V$, where $V$ is some arbitrary one-dimensional subspace. Given a $k$-space $U$ (for $1 \leq k \leq v$) we can describe it by the set $P_U$ of one-dimensional (i.e. numbers between $0$ and $l$) subspaces contained in $U$. Given such a description of a $k$-space $U$ it is now easy to get all the spaces building the orbit under the Singer subgroup $G$. Take the set $P_U$ of $\begin{bmatrix} k \\ 1 \end{bmatrix}_q$ numbers from $\{0,\ldots,l\}$ representing the one-dimensional subspaces of $GF(q)^v$ being also subspaces of $U$ and now the action of $S$ on a number in $P_U$ is simply adding one modulo $l$. If we look at $GF(q)^v$ as an field extension of degree $v$ of the base field $GF(q)$ this is the use of a primitive element $\omega$ and writing the elements of $GF(q)$ using $\omega$ and an exponent.

*Example 1:* We study the case $q = 2, v = 5, k = 2$ : A two-dimensional binary subspace contains three one-dimensional subspaces. We get a two-dimensional space by taking the two one-dimensional spaces labeled $\{0, 1\}$ and the third one given by the linear combination of these two will have a certain number, in this example $\{14\}$. Therefore we have a two dimensional space described by the three numbers $\{0, 1, 14\}$. To get the complete orbit under the Singer subgroup we simply have to increase the numbers by one for each multiplication by a generator $S$ of the Singer subgroup. The orbit length of the Singer subgroup is 31 and the orbit is built by the 31 sets: $\{0, 1, 14\}, \{1, 2, 15\}, \ldots, \{16, 17, 30\}, \{0, 17, 18\}, \ldots$ $\{12, 29, 30\}$, $\{0, 13, 30\}$.

## III. Real Coding

In this chapter we will restrict to the case of a constant dimension subspace code built from a single orbit $O$ of the Singer cycle on the $3-$dimensional subspaces of $GF(2)^v$ with the property that two subspaces of the orbit intersect in a subspace of dimension less or equal to one. We will call such an orbit a *good* orbit. The subspaces in a good orbit $O$ form a code of minimum distance at least $4$. This is a less restrictive version of the codes studied in [8] where a code was studied, where the subspaces intersect zero-dimensional.

We describe a single subspace $U$ in the orbit $O$ by the 7 one-dimensional subspaces contained in $U$. As an example lets assume $U = \{0, 1, 4, 10, 18, 23, 25\}$. The orbit-type can be described by the two distance between $v_1, v_2 \in U$, i.e. the two exponents $i_1, i_2$ of the generator $S$ defined by $S^{i_1} v_1 = v_2$ and $S^{i_2} v_2 = v_1$. In this example it is visualized by the complete graph $K_V$ with vertices from $V$ and edge-labels given by the the smaller distance. For this concrete example we get:



If we take a different subspace from $O$ the vertex labels are increased by a fixed number, but the pairwise distances remain unchanged. We denote by $K_O$ the complete graph with 7 points without vertex-labels and edge-labels given by any $V \in O$. The following observation is crucial for the proposed algorithm for decoding:

*Lemma 1:* All edge-labels in $K_O$ are different.

This is because otherwise we would have no orbit where two subspaces intersect at most one-dimensional. This property is similar to the questions studied in the theory of finite difference sets or Golomb rulers (e.g. [2] p.419ff). Above labeling comes from an optimal Golomb ruler. It is not clear whether there is a primitive element in $GF(2^v)$ such that we get a vector space using above exponents.

If we work with the field $GF(2^v)$ instead of the vector space $GF(2)^v$, than above graph is constructed using the elements $\{v_1, \ldots, v_7\}$ of a subspace as labels of the vertices and the two quotients $v_i/v_j$ and $v_j/v_i$ as labels of the edges.

### A. Decoding

There are two cases, which have to be checked for decoding.

*1) Erasure Case:* The first one is an erasure, meaning that we receive only a two-dimensional subspace $U$, represented by two of the three one-dimensional subspaces $U = \{r, s\}$. The idea now is to identify the two vertices in $K_O$ corresponding to these two elements $r$ and $s$. To do this we only have to compute the quotient (in the field $GF(2^v)$ ) of the two field elements $r$ and $s$. And using the result we can lookup the corresponding edge in $K_O$ and get vertex labels by taking $r$ and $s$. After that we use one further application of some power of the Singer cycle (i.e. multiplication in $GF(2^v)$) to get a third independent vector, which finished the reconstruction of the transmitted vectorspace.

Therefore decoding including error correction in the erasure case is one division and one multiplication in $GF(2^v)$.

*2) Error Case:* The second case is that we received a vector not in the transmitted space $V$. As we have minimum distance 4 the erroneous codeword we want to correct is a $4-$dimensional space $U$ containing the transmitted codeword $V$. Again the idea is to identify the 3-dimensional subspace $V$ inside $U$ by looking at the pairwise distances. We start with 3 independent vectors $r, s, t$ in $U$. We know that the space $I$ generated by $\{r, s, t\}$ intersects with $V$ in a at least two dimensional space $W$. Now we loop over all 7 2-dimensional subspaces of $I$ and for each such space we try the reconstruction like in the erasure case. We additionaly check whether the constructed third vector is in the received word $U$. If this is the case we have found the original codeword $V$.

Therefore decoding including error correction in the error case needs at most 7 divisions and 7 multiplications in $GF(2^v)$.

### B. Generalisations

There are several immediate generalisations. Of course one can use $n$ different orbits of the Singer cycle with the pairwise intersection property for all spaces in the $n$ orbits. In this case you would have $n \cdot 21$ possible pairs of distances (=quotients). One can also use $k-$dimensional subspaces instead of $3-$dimensional subspaces. In this case one has to look at $\left(2^k - 1\right)\left(2^k - 2\right)/2$ pairs of distances and would get a code which allows to correct $k-2$ errors. In the non-erasure case error correction becomes more difficult as one has to study more two-dimensional subspaces.

## IV. How to find a code

In this section we give some arguments showing, that it is 'easy' to find *good* orbits. We study the problem for arbitrary $k$ (not only $k = 3$) and for arbitrary finite fields (not only $q = 2$). For the computation of an estimation how 'easy' it is we use a primitive element $\omega$. For a 'real' application we represent the field elements by polynomials, as the internal nodes of the network have to compute linear combinations, meaning doing exor operations in the case of $q = 2$.

We need to find one *good* orbit $O$ of the Singer cycle such that all $k$-dimensional (in the above special case we had $k = 3$) subspaces intersect pairwise at most in a one-dimensional subspace. To do so, we choose arbitrarily $k$ representatives $a_1, a_2, \ldots, a_k$ of one-dimensional subspaces generating a $k-$space $U$ represented by the one-dimensional subspace $P_U = \{b_0, \ldots, b_l\}$(with $l = \begin{bmatrix} k \\ 1 \end{bmatrix}_q - 1$). If the $\binom{l+1}{2}$ quotient pairs built form $b_i, b_j$ for $i > j$ are all different (meaning that the subspaces generated from the quotient are different), the intersection of all elemens of the orbit of this space under the Singer cycle is at most one-dimensional. Thus, the elements of the orbit $O$ form a code with minimum distance $2(k-1)$. In order to generate such a orbit, we first select a primitive element $\omega$. Then we choose randomly the $k$ representatives of one-dimensional subspaces $a_1, a_2, \ldots, a_k$ by choosing $k$ random numbers $e_i$ between 0 and $q^v - 2$ and setting $a_i = \omega^{e_i}$, $i = 1, \ldots, k$. The subspace $\langle a_1, a_2, \ldots, a_k \rangle$ will be the generator of the orbit $O$. The probability that the dimension of $\langle a_1, a_2, \ldots, a_k \rangle$ is less than $k$ is very small. In practice, for $k = 3$ and $q = 2$ we choose $e_1 = 0$, $e_2 = 1$ and a random number $1 < e_3 \leq 2^v - 2$.

The property that all pairwise differences (mod $q^v - 1$) between all $\begin{bmatrix} k \\ 1 \end{bmatrix}_q$ non-zero elements of the subspace generated by $a_1, a_2, \ldots, a_k$ are different seems to be randomly distributed. For example, for two generators $a_i$ and $a_j$ the integer number $0 \leq x \leq q^v - 2$ with $\omega^x = a_i + a_j$ is likely to be hard to compute. It is called the discrete logarithm problem in cryptography. This problem is also closely related to the problem of computing the Jacobi logarithm [5], also called Zech's logarithm. There are no efficient (polynomial time) algorithms known to compute the discrete logarithm [9], also the discrete logarithm of the sum of arbitrary elements with known discrete logarithm as in our case seems to be without structure.

The overall number $m$ of possible pairs $\{b_i, b_j\}$ (or pairs $\{e_i - e_j \bmod q^v - 1, e_j - e_i \bmod q^v - 1\}$) is equal to $(q^v - 2)/2$ for $q$ even, and $(q^v - 1)/2$ for $q$ odd. Let $s$ be the number of (unordered) pairs of one-dimensional subspaces in a $k$-dimensional vector space. The probability that all $s$ pairs $\{b_i, b_j\}$ are different is equal to

$$(1 - \frac{1}{m})(1 - \frac{2}{m}) \cdots (1 - \frac{s}{m})$$

which is approximately

$$\left(1 - \frac{s}{2m}\right)^{s-1} \approx e^{-s(s-1)/2m}.$$

For example, for $q = 2$, $v = 100$ and $k = 3$ we get $s = \binom{2^3-1}{2} = 21$, $m = 633825300114114700748351602687$ and therefore

$$-s(s-1)/2m \approx -3.3132158019282496 \cdot 10^{-28}.$$

So, it is extremely unlikely to find a random orbit which does not fulfill the desired property.

If we take the union of $n$ orbits as our code we can apply the same estimation. For $q = 2$ we get

$$s = \binom{n \cdot (2^k - 1)}{2}.$$

In the example above, the expected number of orbits which can be combined without conflicts is $66\,955\,225\,653\,132$.

## REFERENCES


[1] Michael Braun, Adalbert Kerber, and Reinhard Laue. Systematic construction of $q$-analogs of $t$-$(v, k, \lambda)$-designs. *Des. Codes Cryptography*, 34(1):55–70, 2005.

[2] Charles J. (ed.) Colbourn and Jeffrey H. (ed.) Dinitz. *The CRC handbook of combinatorial designs. 2nd ed.* Discrete Mathematics and its Applications. Boca Raton, FL: Chapman & Hall/CRC. 984 p., 2007.

[3] Keldon Drudge. On the orbits of Singer groups and their subgroups. *Electronic Journal Comb.*, 9, 2002.

[4] T. Etzion and A. Vardy. A New Construction for Constant Weight Codes. *ArXiv e-prints*, April 2010.

[5] C. G. J. Jacobi. Über die Kreistheilung und ihre Anwendung auf die Zahlentheorie. *J. Reine Angew. Math.*, 30:166–182, 1846.

[6] R. Koetter and F. R. Kschischang. Coding for errors and erasures in random network coding. *Information Theory, IEEE Transactions on*, 54(8):3579–3591, July 2008.

[7] Axel Kohnert and Sascha Kurz. Construction of large constant dimension codes with a prescribed minimum distance. Calmet, Jacques (ed.) et al., Mathematical methods in computer science. Essays in memory of Thomas Beth. Berlin: Springer. Lecture Notes in Computer Science 5393, 31-42 (2008)., 2008.

[8] Felice Manganiello, Elisa Gorla, and Joachim Rosenthal. Spread codes and spread decoding in network coding. *arxiv*, abs/0805.0507, 2008.

[9] Andrew Odlyzko. Discrete logarithms: The past and the future. *Des. Codes Cryptography*, 19(2-3):129–145, 2000.

[10] Simon Thomas. Designs over finite fields. *Geom. Dedicata*, 24:237–242, 1987.



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