Simple 8-(40,11,1440) Designs

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Abstract: In this short note simple 8-(40,11,1440) designs with automorphism group PSL(4,3) are presented. The designs are constructed with the method of Kramer and Mesner on a computer using the software package DISCRETA [2].

A simple t-(v,k,λ) design $\mathcal{D} = (\mathcal{B}, \mathcal{V})$ is a set $\mathcal{B}$ of k-subsets of a v-set $\mathcal{V}$ such that each t-subset of $\mathcal{V}$ is contained exactly λ times in $\mathcal{B}$. A recent overview on existence results of t-designs can be found in [6]. In this paper, we show the existence of 8-(40,11,1440) designs with group of automorphisms PSL(4,3). The construction follows the method of Kramer and Mesner [5] and is done by computer with the software package DISCRETA [2]. There exist more than 100000 designs with this set of parameters and this group of automorphisms. We were not able to construct all designs, yet. Further, determining the number of non-isomorphic designs among the solutions is not yet completed.

In a first step we constructed the Kramer-Mesner matrix $A_{t,k}$ for 8-(40,11,λ) designs by prescribing the group PSL(4,3) as group of automorphisms. For the definition of Kramer-Mesner matrices see [5].

The group $\mathrm{PSL}(4,3)$, whose order is equal to 6065280, is generated by the following permutations, which can be found in the list presented in the CRC Handbook of Combinatorial Designs [3, p. 603]:

(1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39)(2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40),
(1 29 26 4 13)(2 28 38 30 22)(3 8 39 16 32)(5 7 27 14 15)(6 25 23 37 9)(10 17 11 40 35)(12 34 18 24 31)(19 36 21 33 20)

This group has the following number of orbits on s-subsets of $\mathcal{V} = \{1, 2, \dots, 40\}$:

s	0	1	2	3	4	5	6	7	8	9	10	11
# s-orbits	1	1	1	2	4	6	12	24	53	111	263	569

Thus, the Kramer-Mesner matrix $A_{t,k}$ for 8-$(40, 11, \lambda)$ designs using PSL(4,3) as prescribed group of automorphisms has 53 rows and 569 columns. Now, the theorem of Kramer-Mesner [5] tells that each 8-(40,11,1440) design with automorphism group PSL(4,3) corresponds to a solution of the diophantine linear system

$$A_{t,k} \cdot x = (\lambda, \lambda, \dots, \lambda)^\top, \quad \text{where } x \in \{0,1\}^{569}. \tag{1}$$

In a second step this system (1) was solved by an explicit enumeration algorithm based on lattice basis reduction as described in [9]. This algorithm quickly found more than 100000 solutions. Here, we list the orbit representatives of the first solution of (1), given by the algorithm. For each orbit we give the lexicographically minimal representative, where the subscript gives the stabilizer order of the orbit. Each representative consists of an intervall $1, \dots, i$ and some further points. Thus, only i and those further points are listed. As an example, the full base block of the first orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 17\}_2$ is represented as $\{\dots, 10, 17\}_2$. Its stabilizer order is equal to 2, which means that the length of this orbit is $\frac{6065280}{2} = 3032640$.

The base blocks of the selected 8-(40,11,1440) design

$\{\dots,10,17\}_2$
$\{\dots,10,19\}_1$
$\{\dots,10,20\}_1$
$\{\dots,10,35\}_2$
$\{\dots,9,11,16\}_1$
$\{\dots,9,11,19\}_1$
$\{\dots,9,11,20\}_1$
$\{\dots,9,11,23\}_2$
$\{\dots,9,11,24\}_2$
$\{\dots,9,11,30\}_1$
$\{\dots,9,11,33\}_1$
$\{\dots,9,11,34\}_2$
$\{\dots,9,11,37\}_1$
$\{\dots,9,12,16\}_2$
$\{\dots,9,14,17\}_1$
$\{\dots,9,14,18\}_1$
$\{\dots,9,14,26\}_1$
$\{\dots,9,14,34\}_1$
$\{\dots,9,14,36\}_1$
$\{\dots,9,15,19\}_1$
$\{\dots,9,15,24\}_1$
$\{\dots,9,15,33\}_2$
$\{\dots,9,15,35\}_1$
$\{\dots,9,15,38\}_2$
$\{\dots,9,15,39\}_1$
$\{\dots,9,16,17\}_2$
$\{\dots,9,16,18\}_2$
$\{\dots,9,17,23\}_2$
$\{\dots,9,17,35\}_2$
$\{\dots,9,18,27\}_2$
$\{\dots,9,19,25\}_4$
$\{\dots,9,19,34\}_1$
$\{\dots,9,19,40\}_1$
$\{\dots,9,20,21\}_2$
$\{\dots,9,20,34\}_2$
$\{\dots,9,20,36\}_2$
$\{\dots,9,24,25\}_4$
$\{\dots,9,26,40\}_4$
$\{\dots,9,26,33\}_1$
$\{\dots,9,27,34\}_2$
$\{\dots,9,27,37\}_1$
$\{\dots,9,29,35\}_2$
$\{\dots,9,29,37\}_2$
$\{\dots,9,37,38\}_8$
$\{\dots,8,10,11,12\}_2$
$\{\dots,8,10,12,15\}_1$
$\{\dots,8,10,12,16\}_1$
$\{\dots,8,10,12,19\}_1$
$\{\dots,8,10,12,26\}_2$
$\{\dots,8,10,12,34\}_2$
$\{\dots,8,10,14,22\}_1$
$\{\dots,8,10,14,34\}_1$
$\{\dots,8,10,15,25\}_2$
$\{\dots,8,10,16,26\}_1$
$\{\dots,8,10,17,19\}_2$
$\{\dots,8,10,17,26\}_2$
$\{\dots,8,10,17,39\}_2$
$\{\dots,8,11,12,19\}_6$
$\{\dots,8,11,15,16\}_2$
$\{\dots,8,11,15,39\}_2$
$\{\dots,8,11,20,21\}_6$
$\{\dots,8,14,15,27\}_1$
$\{\dots,8,14,15,33\}_1$
$\{\dots,8,14,16,20\}_1$
$\{\dots,8,14,16,26\}_1$
$\{\dots,8,14,16,27\}_1$
$\{\dots,8,14,17,27\}_3$
$\{\dots,8,14,19,20\}_2$
$\{\dots,8,14,19,26\}_2$
$\{\dots,8,14,19,28\}_2$
$\{\dots,8,14,20,24\}_2$
$\{\dots,8,14,20,26\}_2$
$\{\dots,8,14,20,39\}_1$
$\{\dots,8,14,24,35\}_8$
$\{\dots,8,14,26,28\}_2$
$\{\dots,8,15,16,27\}_2$
$\{\dots,8,15,16,33\}_1$
$\{\dots,8,15,17,20\}_2$
$\{\dots,8,15,17,33\}_6$
$\{\dots,8,15,17,37\}_6$
$\{\dots,8,15,20,37\}_6$
$\{\dots,7,9,11,13,17\}_1$
$\{\dots,7,9,11,13,19\}_2$
$\{\dots,7,9,11,13,27\}_2$
$\{\dots,7,9,11,14,19\}_2$
$\{\dots,7,9,11,15,19\}_2$
$\{\dots,7,9,11,16,28\}_{12}$
$\{\dots,7,9,11,17,19\}_2$
$\{\dots,7,9,11,19,22\}_8$
$\{\dots,7,9,13,15,19\}_1$
$\{\dots,7,9,13,15,22\}_1$
$\{\dots,7,9,13,15,25\}_1$
$\{\dots,7,9,13,15,26\}_2$
$\{\dots,7,9,13,15,27\}_1$
$\{\dots,7,9,13,15,39\}_1$
$\{\dots,7,9,13,17,27\}_2$
$\{\dots,7,9,13,17,33\}_1$
$\{\dots,7,9,13,17,34\}_2$
$\{\dots,7,9,13,17,35\}_1$
$\{\dots,7,9,13,17,38\}_1$
$\{\dots,7,9,13,19,27\}_1$
$\{\dots,7,9,13,19,33\}_1$
$\{\dots,7,9,13,19,38\}_6$
$\{\dots,7,9,13,20,32\}_6$
$\{\dots,7,9,13,20,38\}_2$
$\{\dots,7,9,13,22,27\}_1$
$\{\dots,7,9,14,15,35\}_1$
$\{\dots,7,9,14,35,38\}_{12}$
$\{\dots,7,9,15,16,17\}_1$
$\{\dots,7,9,15,16,19\}_2$
$\{\dots,7,9,15,19,26\}_1$
$\{\dots,7,9,15,25,28\}_2$
$\{\dots,7,9,16,19,20\}_{12}$
$\{\dots,7,9,17,19,22\}_1$
$\{\dots,6,8,9,10,11,16\}_1$
$\{\dots,6,8,9,10,11,18\}_1$
$\{\dots,6,8,9,10,11,19\}_1$
$\{\dots,6,8,9,10,11,25\}_1$
$\{\dots,6,8,9,10,11,31\}_1$
$\{\dots,6,8,9,10,12,15\}_3$
$\{\dots,6,8,9,10,12,18\}_1$
$\{\dots,6,8,9,10,12,25\}_1$
$\{\dots,6,8,9,10,12,32\}_1$
$\{\dots,6,8,9,10,12,34\}_1$
$\{\dots,6,8,9,10,12,39\}_1$
$\{\dots,6,8,9,10,14,19\}_2$
$\{\dots,6,8,9,10,14,23\}_1$
$\{\dots,6,8,9,10,14,30\}_1$
$\{\dots,6,8,9,10,14,39\}_1$
$\{\dots,6,8,9,10,15,33\}_1$
$\{\dots,6,8,9,10,16,28\}_2$
$\{\dots,6,8,9,10,17,18\}_2$
$\{\dots,6,8,9,10,17,29\}_1$
$\{\dots,6,8,9,10,20,39\}_2$
$\{\dots,6,8,9,10,23,31\}_1$
$\{\dots,6,8,9,10,25,31\}_2$
$\{\dots,6,8,9,10,29,39\}_3$
$\{\dots,6,8,9,10,30,31\}_1$
$\{\dots,6,8,9,10,31,32\}_1$
$\{\dots,6,8,9,11,15,25\}_1$
$\{\dots,6,8,9,11,15,26\}_2$
$\{\dots,6,8,9,11,15,32\}_1$
$\{\dots,6,8,9,11,17,29\}_3$
$\{\dots,6,8,9,11,18,23\}_2$
$\{\dots,6,8,9,11,18,34\}_1$
$\{\dots,6,8,9,11,19,20\}_1$
$\{\dots,6,8,9,11,19,34\}_2$
$\{\dots,6,8,9,11,21,32\}_1$
$\{\dots,6,8,9,11,25,31\}_{12}$
$\{\dots,6,8,9,11,30,38\}_1$
$\{\dots,6,8,9,11,31,32\}_6$
$\{\dots,6,8,9,11,31,38\}_6$
$\{\dots,6,8,9,12,17,20\}_2$
$\{\dots,6,8,9,12,21,38\}_2$
$\{\dots,6,8,9,12,32,37\}_8$
$\{\dots,6,8,9,14,23,32\}_2$
$\{\dots,6,8,9,14,25,26\}_8$
$\{\dots,6,8,9,14,30,37\}_6$
$\{\dots,6,8,9,14,32,33\}_6$
$\{\dots,6,8,9,16,17,20\}_2$
$\{\dots,6,8,9,16,17,37\}_6$
$\{\dots,6,8,9,17,18,29\}_2$
$\{\dots,6,8,9,17,29,34\}_8$
$\{\dots,6,8,9,18,29,34\}_8$
$\{\dots,6,8,9,19,21,25\}_8$
$\{\dots,6,8,10,14,28,32\}_{24}$
$\{\dots,6,8,11,12,19,21\}_4$
$\{\dots,6,8,11,12,19,24\}_6$
$\{\dots,6,10,11,14,15,26\}_6$
$\{\dots,6,10,11,14,15,30\}_{12}$
$\{\dots,6,10,11,18,19,20\}_{36}$
$\{\dots,6,10,11,18,19,23\}_{12}$
$\{\dots,6,10,11,18,20,23\}_{36}$
$\{\dots,6,8,17,24,27,29\}_{24}$
$\{\dots,6,8,19,21,28,29\}_{12}$
$\{\dots,6,10,14,15,16,26\}_4$
$\{\dots,5,8,9,16,18,34,38\}_{108}$
$\{\dots,4,6,9,10,17,18,22,35\}_{108}$

This design consists of 178 orbits and the number of blocks equals $b = 671\,168160$.

Finally, we give some information on intersection numbers for this specific parameter set. These can be used as invariants to distinguish at least some different isomorphism types, see below. A more extensive treatment of intersection numbers in that context can be found in [1].

For subsets I and J of $\mathcal{V}$ with $|I| = i$, $|J| = j$ and $0 \le i + j \le t$ define

$$\lambda_{i,j} := |\{B \in \mathcal{B} \mid I \subseteq B \wedge J \cap B = \emptyset\}|.$$

Then the intersection triangle $\lambda_{i,j}$ with $0 \le i+j \le t$ for 8-(40,11,1440) designs is the following:

671 168160	486 596916	349 351632	248 223528	174 427344	121 130100	83 060640	56 188080	37 458720
184 571244	137 245284	101 128104	73 796184	53 297244	38 069460	26 872560	18 729360	
47 325960	36 117180	27 331920	20 498940	15 227784	11 196900	8 143200		
11 208780	8 785260	6 832980	5 271156	4 030884	3 053700			
2 423520	1 952280	1 561824	1 240272	977184				
471240	390456	321552	263088					
80784	68904	58464						
11880	10440							
1440								

For an arbitrary fixed m-subset M of $\mathcal{V}$ the i-th *intersection number* of M with the design $\mathcal{D}$ is defined for $0 \le i \le m$ as

$$\alpha_i(M) := |\{B \in \mathcal{B} \mid |M \cap B| = i\}|.$$

Mendelsohn [7] gave the equations

$$\sum_{i=j}^{k} \binom{i}{j} \alpha_i(M) = \binom{m}{j} \lambda_{j,0} \quad \text{for all } j = 0, 1, \ldots, t.$$

Choosing $m = k$, this results in our 8-(40,11,1440) case in:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ & & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 \\ & & & 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120 & 165 \\ & & & & 1 & 5 & 15 & 35 & 70 & 126 & 210 & 330 \\ & & & & & 1 & 6 & 21 & 56 & 126 & 252 & 462 \\ & & & & & & 1 & 7 & 28 & 84 & 210 & 462 \\ & & & & & & & 1 & 8 & 36 & 120 & 330 \\ & & & & & & & & 1 & 9 & 45 & 165 \end{pmatrix} \begin{pmatrix} \alpha_0(M) \\ \alpha_1(M) \\ \alpha_2(M) \\ \alpha_3(M) \\ \alpha_4(M) \\ \alpha_5(M) \\ \alpha_6(M) \\ \alpha_7(M) \\ \alpha_8(M) \\ \alpha_9(M) \\ \alpha_{10}(M) \\ \alpha_{11}(M) \end{pmatrix} = \begin{pmatrix} 671\,168160 \\ 2030\,283684 \\ 2602\,927800 \\ 1849\,448700 \\ 799\,761600 \\ 217\,712880 \\ 37\,322208 \\ 3\,920400 \\ 237600 \end{pmatrix} \quad (2)$$

Köhler [4] expressed $\alpha_0(M), \alpha_1(M), \ldots, \alpha_t(M)$ in terms of $\alpha_{t+1}(M), \ldots, \alpha_m(M)$. Here, for $M \in \mathcal{B}$, i.e. $m = k$, Köhler's equations read as:

$$\begin{array}{lcrrrr} \alpha_0(M) & = & 10\,051704 & -1\,\alpha_9(M) & -9\,\alpha_{10}(M) & -45\,\alpha_{11}(M) \\ \alpha_1(M) & = & 63\,900936 & +9\,\alpha_9(M) & +80\,\alpha_{10}(M) & +396\,\alpha_{11}(M) \\ \alpha_2(M) & = & 160\,180020 & -36\,\alpha_9(M) & -315\,\alpha_{10}(M) & -1540\,\alpha_{11}(M) \\ \alpha_3(M) & = & 204\,995340 & +84\,\alpha_9(M) & +720\,\alpha_{10}(M) & +3465\,\alpha_{11}(M) \\ \alpha_4(M) & = & 150\,448320 & -126\,\alpha_9(M) & -1050\,\alpha_{10}(M) & -4950\,\alpha_{11}(M) \\ \alpha_5(M) & = & 62\,802432 & +126\,\alpha_9(M) & +1008\,\alpha_{10}(M) & +4620\,\alpha_{11}(M) \\ \alpha_6(M) & = & 16\,532208 & -84\,\alpha_9(M) & -630\,\alpha_{10}(M) & -2772\,\alpha_{11}(M) \\ \alpha_7(M) & = & 2019600 & +36\,\alpha_9(M) & +240\,\alpha_{10}(M) & +990\,\alpha_{11}(M) \\ \alpha_8(M) & = & 237600 & -9\,\alpha_9(M) & -45\,\alpha_{10}(M) & -165\,\alpha_{11}(M) \end{array} \quad (3)$$

The *global intersection numbers* $\alpha_i^{(s)}(\mathcal{D})$ of order s of the design $\mathcal{D}$ are:

$$\alpha_i^{(s)}(\mathcal{D}) := \left| \left\{ \{B_{j_1}, \ldots, B_{j_s}\} \in \binom{\mathcal{B}}{s} \mid |B_{j_1} \cap \ldots \cap B_{j_s}| = i \right\} \right|.$$

In [1] it ws shown that these global intersection numbers satisfy the following system of equations

$$\sum_{i=j}^{k} \binom{i}{j} \alpha_i^{(s)}(\mathcal{D}) = \binom{v}{j}\binom{\lambda_{j,0}}{s} \qquad \text{for all } j = 0, 1, \ldots, t,$$

which is derived from the generalized MENDELSOHN system [7, 8]. The main source of information on higher intersection numbers is [8]. For $s = 1$ these equations are exactly the equations (2). For higher values of s only the right hand side of the system differs. For $s = 2$ the right hand side is equal to

$$\begin{pmatrix} 225233\,349163\,308720 \\ 17033\,271963\,568146 \\ 1119\,873221\,297820 \\ 62\,818368\,939810 \\ 2\,936723\,383440 \\ 111033\,333180 \\ 3262\,986936 \\ 70\,561260 \\ 1036080 \end{pmatrix}.$$

As in (3), $\alpha_0^{(s)}(\mathcal{D}), \ldots, \alpha_t^{(s)}(\mathcal{D})$ can be expressed according to [4] and [8] in terms of $\alpha_{t+1}^{(s)}(\mathcal{D}), \ldots, \alpha_{k-1}^{(s)}(\mathcal{D})$ (note that $\alpha_k^{(s)}(\mathcal{D}) = 0$ for $s > 1$). Here, we give the equations for $s = 2$:

$$\begin{array}{rclrr}
\alpha_0^{(2)}(\mathcal{D}) & = & 3373\,176737\,988720 & -1\,\alpha_9^{(2)}(\mathcal{D}) & -9\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_1^{(2)}(\mathcal{D}) & = & 21444\,269709\,994560 & +9\,\alpha_9^{(2)}(\mathcal{D}) & +80\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_2^{(2)}(\mathcal{D}) & = & 53753\,347846\,598400 & -36\,\alpha_9^{(2)}(\mathcal{D}) & -315\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_3^{(2)}(\mathcal{D}) & = & 68794\,335377\,024400 & +84\,\alpha_9^{(2)}(\mathcal{D}) & +720\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_4^{(2)}(\mathcal{D}) & = & 50486\,399913\,549600 & -126\,\alpha_9^{(2)}(\mathcal{D}) & -1050\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_5^{(2)}(\mathcal{D}) & = & 21077\,046762\,932160 & +126\,\alpha_9^{(2)}(\mathcal{D}) & +1008\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_6^{(2)}(\mathcal{D}) & = & 5547\,015572\,978880 & -84\,\alpha_9^{(2)}(\mathcal{D}) & -630\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_7^{(2)}(\mathcal{D}) & = & 678\,077836\,207200 & +36\,\alpha_9^{(2)}(\mathcal{D}) & +240\,\alpha_{10}^{(2)}(\mathcal{D}) \\
\alpha_8^{(2)}(\mathcal{D}) & = & 79\,679406\,034800 & -9\,\alpha_9^{(2)}(\mathcal{D}) & -45\,\alpha_{10}^{(2)}(\mathcal{D})
\end{array}$$

For the above design, DISCRETA computed the following values:

$$\alpha_9^{(2)}(\mathcal{D}) = 2\,169140\,968800, \quad \alpha_{10}^{(2)}(\mathcal{D}) = 30924\,644400\,.$$

These global intersection numbers $\left(\alpha_9^{(2)}(\mathcal{D}), \alpha_{10}^{(2)}(\mathcal{D})\right)$ can be used as a fingerprint of a design. If for two designs $\mathcal{D}_1, \mathcal{D}_2$

$$\left(\alpha_9^{(2)}(\mathcal{D}_1), \alpha_{10}^{(2)}(\mathcal{D}_1)\right) \neq \left(\alpha_9^{(2)}(\mathcal{D}_2), \alpha_{10}^{(2)}(\mathcal{D}_2)\right),$$

then the design are nonisomorphic. Among the first 400 solutions computed by DISCRETA there are 389 different values of $\left(\alpha_9^{(2)}(\mathcal{D}), \alpha_{10}^{(2)}(\mathcal{D})\right)$. So, there exist at least 389 nonisomorphic 8-(40,11,1440) designs.

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