A New Smallest Simple 6-Design With Automorphism Group A_4

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Abstract

A new simple 6-(14,7,4) design is presented with automorphism group isomorphic to A_4. Combining the derived and the residual designs of the 6-(14,7,4) designs, which were known by now, in the extension method of van Leijenhorst and Tran van Trung results in a large number of simple 5-(14,7,18) designs with trivial automorphism group. This parameter set results from interpreting a 6-(14,7,4) design as a 5-design.

keywords: t-design, isomorphism problem, double coset.

1 Introduction

For almost two decades, only two isomorphism types of 6-$(14,7,4)$ designs have been known, both with automorphism group C_{13} acting with an additional fixed point. They were found by Kreher and Radziszowski [4] using a lattice basis reduction in Kramer and Mesner's approach [3] of combining orbits of a prescribed automorphism group. This parameter set deserves special interest as it is the smallest admissible parameter set of a 6-design. Each such design consists of half of all possible 6-sets on 14 points. Thus, together with its complement it forms a large set and is a starting point for various recursive constructions of infinite series of t-designs. Recently, Eslami and Khosrovshahi [2] constructed 4 further 6-$(14, 7, 4)$ designs using trades and determined the possible derived designs of the desired smallest 6-designs. These new 6-designs only admit C_3 as full automorphism group.

In this note, we present a simple 6-$(14, 7, 4)$ design with automorphism group G isomorphic to A_4. In order to obtain a permutation representation of A_4 on a set of 14 points, we consider the action of A_4 on the set of vertices of a solid that is derived from the tetrahedron. The solid is constructed in two extension steps. In the first step the dual tetrahedron is inscribed by taking the centers of the faces as new vertices. In the second step, an octahedron is inscribed by taking the midpoints of the edges of the first tetrahedron, see Fig. 1 below. No automorphism is admitted that would interchange the two tetrahedra, since the two have different sizes.

Theorem 1 *There exists exactly one isomorphism type of simple 6-(14,7,4) designs with full automorphism group G.*

Proof Prescribing this permutation group our software package DISCRETA yields 8 solutions to the Kramer-Mesner system of diophantine equations. The 8 designs are isomorphic, since they are already in only one orbit under the action of the normalizer of G in S_{14} which has order 96. We will also deduce that G is the full automorphism group of each of these 8 designs. So, we fix one of these designs $\mathcal{D}$. From the discussion in [2] it is clear that non-trivial automorphisms of $\mathcal{D}$ have order 3 or some power of 2. If a group of automorphisms of order 9 would exist then this group would have two orbits of length 1. The design derived at the two fixed points then also would admit this group. From [2] we know that each non-identity automorphism of a simple 4-$(12, 5, 4)$ design has no fixed points. But then each orbit would have length 9 which is impossible on 12 points. So the index of G in the full automorphism group is some power of 2 and by Burnside's Theorem on $p^\alpha q^\beta$-groups $Aut(\mathcal{D})$ is soluble. Suppose G is not the full automorphism group. From $G = N_{S_{14}}(G) \cap Aut(\mathcal{D})$ we obtain

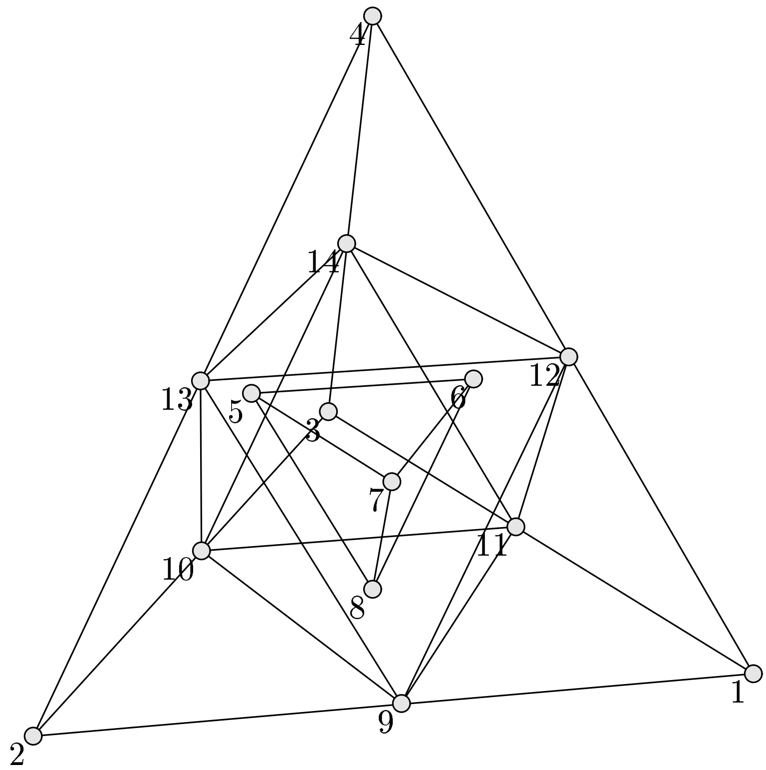

Figure 1:

that $N_{Aut(\mathcal{D})}(G) = G$. So, if G is a maximal subgroup of a subgroup H of $Aut(\mathcal{D})$ then the index of G in H must be at least 4. Then we consider a subgroup Q of order 3 of G. This subgroup is not normal in G and also not in H. Therefore the Fitting subgroup of $F(H)$ is a 2-group not containing G. Thus, $H = F(H)G$ and $F(H) \cap G = F(G) = V_4$. The factor $F(H)/F(G)$ is a chief factor of H on which Q acts irreducibly and therefore is elementary abelian of order 4. So we know some structure of H. The subgroup Q is a Sylow-3 subgroup which is a complement of $F(H)$ in H. If $Q < N_H(Q)$ then $N_H(Q)$ has order 12 and Q acts trivially on $N_H(Q) \cap F(H)$. Neither $F(G)$ nor $F(H)/F(G)$ allow any non-trivial fixed points of Q under conjugation. Therefore $Q = N_H(Q)$.

Consider a point x fixed by Q. The stabilizer S of x in H cannot be Q, as otherwise the orbit of x would have length 16. So $Q < S < H$ and $S \cap F(H)$ is normalized by Q. Like in the case of the normalizer Q acts non-trivially on $S \cap F(H)$. The order of $S \cap F(H)$ thus has to be at least 4. If $S \cap F(H)$ is not normal in $F(H)$ then $S \cap F(H) < N_{F(H)}(S \cap F(H)) < F(H)$ and Q leaves $N_{F(H)}(S \cap F(H))$ invariant. Then Q acts trivially on $F(H)/N_{F(H)}(S \cap F(H))$ and on $N_{F(H)}(S \cap F(H))/S \cap F(H)$ because both are of order 2. Thus, $S \cap F(H)$ is normal in $F(H)$ and Q acts non-trivially on $F(H)/S \cap F(H)$. A check with DISCRETA shows that the group of order 48 constructed as a subdirect product of two copies of A_4 with amalgamated factor group of order 3 acting with two orbits of length 4 and one orbit of length 6 is not a group of automorphisms of a 6-(14,7,4) design. The other potential group of order 16 admitting an automorphism group of order 3 acting in the described way is an extension of $C_4 \times C_4$ by C_3. This group has no faithful action on 14 points. This proves our claim. □

We point out that the non-abelian group A_4 has 3 orbits, two of length 4 and one of length 6, on the set of vertices in this action but no fixed points. This can be deduced from our presentation as the automorphism group of nested solids. The vertices of the tetrahedra form the two orbits of length 4 and the vertices of the octahedron form the orbit of length 6. Thus, there result 3 isomorphism types of derived designs with parameters 5-$(13, 6, 4)$, two with automorphism group of order 3 and one with automorphism group of order 2. Notice that each automorphism of a 5-$(13, 6, 4)$ design extends to an automorphism of the 6-(14,7,4) design obtained by Alltop's construction.

The approach presented in [5] implies that a large number of isomorphism types of 5-$(14, 7, 18)$ designs exist, most of them with trivial automorphism group: We combine the different isomorphism types of 5-$(13, 6, 4)$ designs with the different isomorphism types of 5-$(13, 7, 14)$ designs which appear as derived and residual designs of the by now known simple 6-$(14, 7, 4)$ designs using the construction of van Leijenhorst [7] and Tran van

Trung [6]. We use the following notation. If $\mathcal{D}$ is a t-(v,k,λ) design with point set $\{1,\ldots,v\}$ then $\mathcal{D}*\{v+1\}$ denotes the set of blocks of $\mathcal{D}$ extended by an additional point $v+1$. Any permutation π on the point set maps $\mathcal{D}$ onto an isomorphic design $\mathcal{D}^\pi = \{B^\pi | B \in \mathcal{D}\}$ where B^π denotes the image of B under π. We have the following result [5]:

Theorem 2 *Let $\mathcal{D}_1$ be a $(t-1)$-$(v-1,k-1,\lambda)$ design with automorphism group A_1 and $\mathcal{D}_2$ be a $(t-1)$-$(v-1,k,\lambda(v-k)/(k-t+1))$ design with automorphism group A_2, where the point set in each case is $V' = \{1,\cdots,v-1\}$. Then $\mathcal{D}_1 * \{v\} \cup \mathcal{D}_2^\pi$ is a $(t-1)$-$(v,k,\lambda(v-t+1)/(k-t+1))$ design for each permutation π on $V' = \{1,\ldots,v-1\}$. There exists an isomorphism*

$$\phi : \mathcal{D}(\pi_1) \mapsto \mathcal{D}(\pi_2)$$

for permutations π_1, π_2 on V' such that ϕ fixes v if and only if

$$A_1 \pi_1 A_2 = A_1 \pi_2 A_2.$$

Taking as $\mathcal{D}_1$ the derived design of a 6-(14,7,4) design and as $\mathcal{D}_2$ the residual design of a 6-(14,7,4) design then by the Theorem there result many different isomorphism types of 5-$(14,7,18)$ designs.

We consider a special case that is easy to analyse. Suppose $\mathcal{D}_1$ with parameters 5-$(13,6,4)$ and $\mathcal{D}_2$ with parameters 5-$(13,7,14)$ are derived and residual designs, resp., of non-isomorphic 6-(14,7,4) designs. Then Alltop's construction in each case reconstructs the original 6-(14,7,4) designs. So, if the reconstructed designs are non-isomorphic then the designs resulting from van Leijenhorst's and Tran van Trung's construction are 5-$(14,7,18)$ designs but no 6-(14,7,4) designs.

This situation appears very often. If we start with a 5-$(13,6,4)$ design and apply Alltop's construction we obtain a 6-$(14,7,4)$ design. Each automorphism of the 5-$(13,6,4)$ design then also extends to an automorphism of the 6-$(14,7,4)$ design fixing the additional point. Thus, each automorphism also is an automorphism of the residual design. The same holds true, vice versa, if we start with the residual design and transfer the automorphisms to the derived design. Therefore, if in a 5-$(14, 7, 18)$ design the derived design and the residual design with respect to some point are 5-designs with different automorphism groups, the 5-$(14, 7, 18)$ design cannot be a 6-$(14, 7, 4)$ design.

We obtain the following cases of 5-$(13, 6, 4)$ designs:

1. The construction by Kreher and Radziszowski yields 2 isomorphism types with automorphism group C_{13}, and 2 isomorphism types with trivial automorphism group.

2. The construction by Eslami and Khosrovshahi yields 1 isomorphism type with automorphism group $C_3 \times Id_4+$ and 8 isomorphism types with trivial automorphism group, see [2].

3. The new construction of this paper yields 2 isomorphism type with automorphism group $C_3 \times Id_4+$ and 1 isomorphism type with automorphism group $C_2 \times Id_6+$.

These designs from the different origins are not isomorphic because their Alltop extensions have different automorphism groups. By [5], Theorem 23, the new point 14 is unique in all these designs and the isomorphism types are in bijection to the double cosets $A \backslash S_{13} / C_{13}$. The resulting designs all have trivial automorphism group because no non-trivial subgroup of C_{13} is conjugate to a subgroup of A. Each double coset in $A \backslash S_{13} / C_{13}$ consists of just 13 right cosets of A in S_{13}. This yields the following result:

Corollary 1 *Let $\mathcal{D}_1$ be a* 5-$(13, 6, 4)$ *design with automorphism group A different from C_{13} and $\mathcal{D}_2$ a* 5-$(13, 7, 14)$ *design with automorphism group C_{13}. Then there exist* $12!/|A|$ *different isomorphism types of 5-(14,7,18) designs of the form $\mathcal{D}_1 * \{14\} \cup \mathcal{D}_2^{\pi}$ where $\pi \in S_{13}$. Each of these designs has a trivial automorphism group.*

For $Aut(\mathcal{D}_1) \cong C_2$ we thus obtain 238,500,800 isomorphism types and for $Aut(\mathcal{D}_1) \cong C_3$ we thus obtain 159,667,200 isomorphism types of 5-(14,7,18) designs with trivial automorphism group.

We present the new 6-(14,7,4) design by a list of canonical representatives from the orbits of G on the set of blocks.

2 The 6-$(14, 7, 4)$ design

The automorphism group is

$$G = \langle (1\,2\,3)(5\,6\,7)(9\,10\,11)(12\,13\,14), \\ (1\,2\,4)(5\,6\,8)(9\,13\,12)(10\,14\,11) \rangle$$

of order 12.

There are 1716 blocks, each point lies in half of the blocks.

The design $\mathfrak{D}$ consists of 152 orbits of G on 7-sets. We list all orbit representatives of blocks, with orbit length and stabilizer order appended.

$\{1, 2, 10, 11, 12, 13, 14\}_{6,2}$	$\{1, 2, 3, 5, 10, 13, 14\}_{12,1}$	$\{1, 2, 3, 5, 6, 7, 8\}_{4,3}$
$\{1, 2, 3, 4, 5, 6, 11\}_{12,1}$	$\{1, 2, 3, 5, 11, 12, 14\}_{12,1}$	$\{1, 2, 3, 5, 6, 8, 11\}_{12,1}$
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$\{1,6,7,10,12,13,14\}_{12,1}$	$\{1,6,7,9,11,13,14\}_{12,1}$	$\{5,9,10,11,12,13,14\}_{4,3}$
$\{1,6,7,8,9,10,12\}_{12,1}$	$\{1,6,9,10,11,13,14\}_{12,1}$	

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