Constructing Objects up to Isomorphism, Simple 9-Designs with small parameters

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Abstract

Group actions are reviewed as a tool for classifying combinatorial objects up to isomorphism. The objective is a general theory for constructing representatives of isomorphism types. Homomorphisms of group actions allow to reduce problem sizes stepwise. In particular, classifying by stabilizer type, i.e. the automorphism group of the objects, is generalized to using only sufficiently large subgroups of stabilizers. So, less knowledge on the full subgroup lattice of the classifying group is needed. For single steps in the homomorphism decomposition, isomorphism problems are transformed into double coset problems in groups. New lower bounds are given for the number of long double cosets such that corresponding bounds for the number of objects with trivial automorphism group are be derived.

The theory is illustrated by an account of recent work on the construction of t-designs including new results. Based on a computer search by DISCRETA several simple 8-designs and the first simple 9-designs with small parameters are presented. The automorphism group is $ASL(3,3)$ acting on 27 and 28 points. There are many isomorphism types in each case. The number of isomorphism types is determined in the smaller cases. From relating the isomorphism types of design extensions to double cosets also designs with small automorphism groups are accessible. There result more than 10^{16} isomorphism types of 8-$(28,14,\lambda')$ designs from each 8-$(27,13,\lambda)$ design. There are exactly 131,210,855,332,052,182,104 isomorphism types of 7-$(25,9,45)$ designs obtained from extending all the 7-$(24,8,5)$ designs with automorphism group $PSL(2,23)$ by all the 7-$(24,9,40)$ designs with automorphism group $PGL(2,23)$. Most of these designs have a trivial automorphism group. Iterating forming extensions then results in more than 10^{62} isomorphism types of 7-$(26,10,342)$ designs.

keywords: Group actions, isomorphism problem, double coset, t-design, Kramer-Mesner method.

1 Introduction

In mathematics, a natural aim is to describe the objects that are considered. Ideally, a Hauptsatz would fully determine some infinite series and maybe some finitely many additional sporadic objects comprising all cases. This has been achieved in algebra for finite fields, finite abelian groups, finite simple groups etc. The results are used to derive further classifications from these.

In combinatorics, the objects usually have a less regular structure to allow such a comprehensive theorem. So, on one hand the weaker aim to only count the objects of a fixed size is pursued. There are ingenious solutions for many cases, some relying on a fairly general method. A prominent example is Pólya's and Redfield's theory of counting [45, 13, 26].

In applications, there is a need of not only knowing abstractly the existence of some number of objects but to really have the objects. This is obviuos when the isomers that are searched for correspond to some given spectra. Also, block-designs can of course be applied in the planning of experiments in agriculture only if they are explicitely known. A code can be used only if it is at hand.

The development of powerful and cheap computers in the last decade now allows to solve such construction problems in many interesting cases. It is even possible to find constructive solutions where an efficient counting method is not available.

It should therefore be a natural task to excerpt from the different algorithmic approaches the common aspects. Like a theory of counting allows to tackle various problems in a similar way, a theory of construction should give general rules applicable in a larger variety of problems.

This has been a motivation for several papers by A. Kerber and the author and some books by A. Kerber [26]. These all rely on implementations of the algorithms and got an inportant stimulus from practical experiences. Many aspects of applications to the construction of isomers, of groups and of codes have already sufficiently been explained in some specialized papers and some review articles, see [35, 25, 36] and the references there. Here we add some material that resulted from the search for t-designs with "large" t on small point sets, where large means $t \geq 3$. The t-designs are combinatorial objects defined on a point set V of v points. We only consider simple t-designs $\mathcal{D}$ which consist of a collection of k-element subsets, called blocks, of V, such that each t-element subset of V lies in exactly the same number λ of blocks. The numeric parameters of $\mathcal{D}$ are listed as t-(v, k, λ). Usually, constructing t-designs and solving isomorphism problems for t-designs are difficult. We use a group-action approach for solving these problems by algebraic means. It is important to notice that isomorphism problems sometimes are easier to solve if information about the way of construction is used. So, we follow up this idea and thus avoid to solve the general isomorphism problem.

We first give a summary of group theoretic methods which form the abstract background. They are collected out of several recent papers. Then, the use of these methods in the search for t-

designs is explained. On one hand, prescribed automorphism groups are used to deal with whole orbits of these groups instead of the individual elements like t-sets, k-sets or even designs. On the other hand, these groups yield a powerfull tool for isomorphism classification. This had already been developed in the recent papers on 6-, 7-, and 8-designs constructed with help of a computer by our system DISCRETA. In this paper we continue with the first simple 9-designs on small point sets and then consider the isomorphism problem for design extensions. We use double cosets which often correspond to the isomorphism types. So, for the first time huge numbers of isomorphism types can be determined. Most of these designs have a trivial automorphism group. The following tables illustrate these results.

The new and earlier results on t-designs with $t \geq 8$ and $v \leq 40$ and big automorphism group are summarized in the following table. All results concerning $ASL(3,3)$ are new.

Simple 8- and 9-designs

Parameters	Group	Size of KM-matrix	Number of isomorphism types
8-(27,11,432)	$ASL(3,3)$	31×121	1
8-(27,12,1296)	$ASL(3,3)$	31×154	4336
8-(27,12,1932)	$ASL(3,3)$	31×154	2110899
8-(27,13,3204)	$ASL(3,3)$	31×176	538218
8-(27,13,3240)	$ASL(3,3)$	31×176	618421
8-(27,13,4608)	$ASL(3,3)$	31×176	$\geq$ 200000000
8-(27,13,5076)	$ASL(3,3)$	31×176	many
8-(27,13,5148)	$ASL(3,3)$	31×176	many
8-(28,13,5832)	$ASL(3,3)+$	48×330	$\geq$ 5000000000
8-(28,13,7080)	$ASL(3,3)+$	48×330	many
8-(28,13,7128)	$ASL(3,3)+$	48×330	many
8-(28,14,10680)	$ASL(3,3)+$	48×352	≥ 1
8-(28,14,10800)	$ASL(3,3)+$	48×352	≥ 1
8-(28,14,14040)	$ASL(3,3)+$	48×352	≥ 1
8-(28,14,15360)	$ASL(3,3)+$	48×352	≥ 1
8-(28,14,16920)	$ASL(3,3)+$	48×352	≥ 1
8-(28,14,17160)	$ASL(3,3)+$	48×352	≥ 1
8-(28,14,18600)	$ASL(3,3)+$	48×352	≥ 1
8-(31,10,93)	$PSL(3,5)$	42×174	138
8-(31,10,100)	$PSL(3,5)$	42×174	1658
8-(36,11,1260)	$Sp(6,2)_{36}$	79×694	≥ 1
8-(40,11,1440)	$PSL(4,3)$	53×569	$\geq$ 150000000
9-(28,14,3204)	$ASL(3,3)+$		538218
9-(28,14,3240)	$ASL(3,3)+$		618421
9-(28,14,4608)	$ASL(3,3)+$		$\geq$ 200000000
9-(28,14,5076)	$ASL(3,3)+$		many
9-(28,14,5148)	$ASL(3,3)+$		many

The parameter list on 27 points and the group $ASL(3,3)$ is complete.

It is remarkable that up to now no 8-design with an automorphism group $PGL(2,q)$ has been found. Also the values of λ for 8-designs are large compared to those of the 7-designs found by prescribing some $PGL(2,q)$, [4]. Since small values of λ are of interest, we also list the few known parameter sets of 6- and 7-designs with $\lambda \leq 10$, omitting derived designs.

Simple 6- and 7-designs

Parameter set	Constructed by	No. of isomorphism types
6-(14,7,4)	$C_{13}+$	2
6-(19,7,4)	$Hol(C_{17})++$	1
6-(19,7,6)	$Hol(C_{19})$	3
6-(22,7,8)	large set recursion	
6-(28,7,6)	$PSU(3,9)$	$\geq$ 10
6-(32,7,6)	$PSL(2,31)$	$\geq$ 18
7-(24,8,4)	$PSL(2,23)$	1
7-(24,8,5)	$PSL(2,23)$	138
7-(24,8,6)	$PSL(2,23)$	$\geq$ 132
7-(24,8,7)	$PSL(2,23)$	$\geq$ 126
7-(24,8,8)	$PSL(2,23)$	$\geq$ 63
7-(26,8,6)	$PGL(2,25)$	$\geq$ 7
7-(33,8,10)	$P\Gamma L(2,32)$	4996426

Without restrictions on λ there are about 400 parameter sets of 7-designs and about 1100 parameter sets of 6-designs with up to 40 points in the database of DISCRETA now.

Using the theory of design extensions presented in this paper, we obtain the following table of lower bounds for the number of isomorphism types of 7-designs.

Extensions of Designs

No.	Parameters	Group	Number of isomorphism types	Parameters	Number of isomorphism types
1	7-(24, 9, 48)	$PGL(2,23)$	$\geq$ 2827		
1	7-(24,10,240)	$PGL(2,23)$	$\geq$ 91	7-(25,10,288)	$\geq$ 23786911342165204970
2	7-(24, 9, 64)	$PGL(2,23)$	$\geq$ 15335		
2	7-(24,10,320)	$PGL(2,23)$	$\geq$ 2	7-(25,10,384)	$\geq$ 5161263324118902274
3	7-(24,8, 5)	$PSL(2,23)$	138		
3	7-(24,9,40)	$PGL(2,23)$	113	7-(25,9,45)	1312108553320521821 04
4	7-(24,8, 6)	$PSL(2,23)$	$\geq$ 132		
4	7-(24,9,48)	$PGL(2,23)$	$\geq$ 2827	7-(25,9,54)	$\geq$ 12559489188663228224 1
5	7-(24,8, 8)	$PSL(2,23)$	$\geq$ 63		
5	7-(24,9,64)	$PGL(2,23)$	$\geq$ 15335	7-(25,9,72)	$\geq$ 7951408124200930620
6	7-(26,8, 6)	$PGL(2,25)$	7		
6	7-(26,9,54)	$P\Gamma L(2,25)$	3989	7-(27,9,60)	$\geq$ 12180277268544144601 8
7	7-(26,12, 5796)	$P\Gamma L(2,25)$	$\geq$ 1		
7	7-(26,13,13524)	$\geq A\Gamma L(1,25)$	$\geq$ 1	7-(27,13,19320)	$\geq$ 1
8	7-(27,10, 540)	$P\Gamma L(2,25)+$	$\geq$ 1		
8	7-(27,11,2295)	$AGL(3,3)$	$\geq$ 105	7-(28,11,2835)	$\geq$ 5754099169659337180
9	7-(27,11, 810)	$ASL(3,3)$	1188		
9	7-(27,12,2592)	$AGL(3,3)$	33	7-(28,12,3402)	$\geq$ 28131151518639117392 4
10	7-(27,11,2025)	$AGL(3,3)$	57		
10	7-(27,12,6480)	$AGL(3,3)$	$\geq$ 500	7-(28,12,8505)	$\geq$ 33743174195947303455 00
11	7-(28,13,10080)	$Sp(6,2)$	1		
11	7-(28,14,21600)	$\geq Sp(6,2)_1$	$\geq$ 1	7-(29,14,31680)	$\geq$ 1
12	7-(25, 9, 54)	Id	$\geq$ 1110672408739331856 0		
12	7-(25,10,288)	Id	$\geq$ 2378690075383402391 6	7-(26,10,342)	$\geq 10^{62}$

The system DISCRETA is freely available from our web-page
http://www.mathe2.uni-bayreuth.de/ discreta/
which also contains an account of the presently known t-designs for $t > 5$, Steiner 5-designs, and further information. The author thanks the DISCRETA research group, in particular Anton Betten and Alfred Wassermann, for their support and Axel Kohnert for computing numbers of double cosets with his system SYMMETRICA.

2 Definitions and Notations

If a group G acts on a set Ω and Δ is a subset of Ω then $N_G(\Delta) = \{g \in G | \{\delta^g | \delta \in \Delta\} = \Delta\}$ is the normalizer of Δ in G. This generalizes the notion of the normalizer in the special case of the conjugation action of a group on its lattice of subgroups. This normalizer acts on Δ and the kernel of this action is $C_G(\Delta)$, the centralizer of Δ in G. We prefer this notion to the setwise and pointwise stabilizers if Δ consists of more than one point. As usual, G_ω is also used to denote the stabilizer of ω in G. For a group element $g \in G$ the set of fixed points is $C_\Omega(g) = \{\omega \in \Omega | \omega^g = \omega\}$.
We assume throughout the paper that $V = \{1, 2, \cdots, v\}$ is a set of natural numbers. Any subset of size k is called a k-set. The symmetric group on V is denoted by S_V.
If A and B are two subgroups of the group G then $A \backslash G / B = \{AgB | g \in G\}$ is the set of double cosets of A and B in G.

3 Group Actions

An important strategy is to transform an isomorphism problem from some family of objects into a group theoretic problem. The following basic results often allow this transfer.

Theorem 1 (Fundamental Lemma) *Let a group G act transitively on a set Ω and $\omega \in \Omega$. Then the mapping $\phi : \Omega \mapsto N_G(\omega) \backslash G$ such that $\phi(\omega^g) = N_G(\omega)g$ is a bijection.*

The action of G on Ω is replaced by right-multiplication on the set of right cosets of $N_G(\omega)$ in G. Restricting the acting group to a subgroup gives a description of the orbits of that subgroup.

Theorem 2 (Split of Orbits) *Let U be a subgroup of G where G acts transitively on Ω. Then*

$$\omega^{gU} \mapsto N_G(\omega)gU$$

defines a bijection between the U-orbits on Ω and the double cosets $N_G(\omega) \backslash G / U$.

There is another situation which leads to double cosets.

Theorem 3 (Gluing Lemma) *Let a group G_1 be a group of automorphisms of some object ω_1 and a group G_2 be a group of automorphisms of some object ω_2. Let $f : \omega_1 \mapsto \omega_2$ be a fixed isomorphism. Then each isomorphism is obtained by composing f with some automorphism α of ω_2, such that the set of all isomorphisms is described by a group;*

$$Iso(\omega_1, \omega_2) = fAut(\omega_2).$$

$G_1 \times G_2$ acts on $Iso(\omega_1, \omega_2)$ by

$$f\alpha^{(g_1,g_2)} = g_1^{-1} f \alpha g_2 = f(f^{-1} g_1^{-1} f)\alpha g_2$$

for $(g_1, g_2) \in G_1 \times G_2$. Thus, the orbits of $G_1 \times G_2$ are in bijection to the double cosets

$$(f^{-1} G_1 f)\alpha G_2$$

in $Aut(\omega_2)$.

This Lemma appears in different applications independently in the literature. An early instance can be found in Ph. Hall's lecture series in Göttingen in 1939, [23], where ω_1 is a factor group of one group and ω_2 a subgroup of another group. The different ways of identifying ω_1 with ω_2 have to be classifyied with respect to equivalence under two groups of automorphisms acting on ω_1 and ω_2, respectively. Such identifications had already earlier ben carried out by Lunn and Senior [41] for a classification of subdirect products of groups. It thus may have been known to these authors before. Other group constructions like semidirect products, central amalgamations etc. are considered by the present author in [29], [36] [25]. In Chemistry, Ruch et al. [48] identified places on a skeleton of a molecule with ligands that should be distributed to these places. This plays an important role in mathematical generators for isomers like the early Dendral [39] and Molgen [21]. We will give a new application to the construction of t-designs below.

Algorithms for solving double coset problems are presented in [14, 30, 17, 40, 50, 51]. In many cases the number of double cosets is very big. Then one can at least count them by combinatorial methods, like the Cauchy Frobenius Lemma. We refer to Kerber's book [26]. So, from an implementation of Redfield's cap-product by H. Fripertinger in A. Kohnert's system SYMMETRICA we obtained the following numbers of double cosets that we will use in our section on designs.

Double Cosets	Number
$PSL(2,23)\backslash S_{24}/PSL(2,23)$	16,828,376,982,435,832
$PSL(2,23)\backslash S_{24}/PGL(2,23)$	8,414,188,491,217,916
$PGL(2,23)\backslash S_{24}/PGL(2,23)$	4,207,094,330,061,055
$PGL(2,25)\backslash S_{26}/PGL(2,25)$	1,657,180,580,754,274,540
$P\Gamma L(2,25)\backslash S_{26}/P\Gamma L(2,25)$	414,295,145,235,066,413
$PGL(2,25)\backslash S_{26}/P\Gamma L(2,25)$	828,590,290,377,152,694
$AGL(1,25)+\backslash S_{26}/P\Gamma L(2,25)$	10,771,673,642,332,865,588
$AGL(3,3)\backslash S_{27}/AGL(3,3)$	118,397,102,441,920,363
$ASL(3,3)\backslash S_{27}/AGL(3,3)$	236,794,204,702,349,473
$ASL(3,3)\backslash S_{27}/ASL(3,3)$	473,588,409,404,698,946
$AGL(3,3)\backslash S_{27}/P\Gamma L(2,25)+$	1,150,819,833,931,867,436
$Sp(6,2)_{28}\backslash S_{28}/Sp(6,2)_{28}$	144,708,746,195,525,184

Usually, in applications most of the orbits of a group are long orbits. The elements in such orbits then have a trivial stabilizer. Usually, it is difficult to determine the number of these orbits. We give at least a lower bound for this number. Orbits different from long orbits are usually called short orbits.

Lemma 1 *Let a group G act on a set Ω and let each $g \in G$, $g \neq id$, have at most c fixed points. Then there are at most $a_s = 2 \cdot c/|G|$ short orbits. The number a_l of long orbits is at least $|\Omega|/|G| - (1 - 1/|G|)c$. If the total number of orbits is a then*

$$a_s \leq 2 \cdot a - 2\frac{|\Omega|}{|G|}.$$

Proof Let $\Omega' = \bigcup_{g \neq id} C_\Omega(g)$, where the union runs over all $g \in G$ different from the identity. Then all short orbits are formed from elements in Ω'. So, counting these orbits by the Cauchy-Frobenius Lemma would require to know the number of fixed points for each group element, including the identity. We use the crude bound $|C_{\Omega'}(id)| = |\Omega'| \leq \sum_{g \neq id} |C_\Omega(g)|$. Then we get for the number a_s of short orbits

$$\begin{aligned} a_s &= \frac{1}{|G|}\sum_{g \in G} |C_{\Omega'}(g)| = \frac{1}{|G|}(|C_{\Omega'}(id)| + \sum_{g \neq id} |C_\Omega(g)|) \\ &\leq \frac{2}{|G|}\sum_{g \neq id} |C_\Omega(g)| \leq \frac{2c}{|G|}. \end{aligned}$$

For the number a_l of long orbits we get

$$\begin{aligned} a_l &= \frac{1}{|G|}\sum_{g \in G} |C_\Omega(g)| - \frac{1}{|G|}\sum_{g \in G} |C_{\Omega'}(g)| = \frac{1}{|G|}(C_\Omega(id) - C_{\Omega'}(id)) \\ &= \frac{1}{|G|}(|\Omega| - |\Omega'|) \geq \frac{1}{|G|}(|\Omega| - \sum_{g \neq id} |C_\Omega(g)|) \\ &\geq \frac{1}{|G|}(|\Omega| - \sum_{g \neq id} c) = \frac{1}{|G|}(|\Omega| - (|G| - 1)c). \end{aligned}$$

From this we obtain the first inequality. The second one is obtained from a combination of the above arguments with the Cauchy-Frobenius Lemma. So, we multiply

$$a = \frac{1}{|G|}(|\Omega| + \sum_{g \neq id} |C_\Omega(g)|)$$

by 2 and use the above bound

$$a_s \leq \frac{2}{|G|}\sum_{g \neq id} |C_\Omega(g)|$$

to get

$$2a - a_s \geq \frac{2|\Omega|}{|G|}$$

which is equivalent to the claimed inequality.

Example. $PGL(2,23)$ has $a = 83$ orbits on 8-sets such that the second equality gives $a_s \leq 44$. Actually, there exist exactly 39 short orbits in this situation. So, the bound seems to be reasonably good. But for our purpose to find designs with trivial automorphism group it is not good enough.

We obtain a sharper bound by considering only points which are fixed by some subgroup of prime order of $PGL(2,p)$, p an odd prime. We are interested in the action of $PGL(2,p)$ by multiplication from the right on the set of right cosets of $PGL(2,p)$ in S_{p+1}, i. e. we investigate the double cosets $PGL(2,p)\backslash S_{p+1}/PGL(2,p)$.

Theorem 4 *$PGL(2,p)$ has at least*

$$\frac{1}{(p+1)p(p-1)}\{(p-2)! - \{\frac{p(p-1)}{4(p+1)}\sum_{d|p+1,dprime}(d-1)d^{(p+1)/d}((p+1)/d)! +$$
$$(p+1) + \frac{p(p+1)}{4(p-1)}\sum_{d|p-1,dprime}(d-1)d^{(p-1)/d}((p-1)/d)!\}\}$$

long double cosets in S_{p+1}.

Proof Let $G = PGL(2,p)$ and let $U \leq G \leq S_{p+1}$. Then $G\pi U = G\pi$ for some $\pi \in S_{p+1}$ if and only if $\pi U \pi^{-1} \leq G$. For some fixed $U' \leq G$ the elements π conjugating U onto U' form a coset of $N_{S_{p+1}}(U)$. So, there are $|N_{S_{p+1}}(U)|$ such elements. We have to multiply this number by the number of choices for U' and then divide by $|G|$, because these elements fall into cosets of G. Then the number of cosets fixed by U is determined. Now, U has $G : N_G(U)$ conjugates in G each of which has this number of fixed points. Lastly we have to sum these numbers of fixed points over all subgroups U of some prime order. Subtracting this number from the number of cosets of G in S_{p+1}, which is $(p-2)!$, gives a lower bound for the number of cosets which are not fixed under any non-trivial element of G. All these cosets then form double cosets consisting of $|G|$ cosets such that dividing by $|G|$ gives a lower bound for the number of long double cosets.

We use some well known results on subgroups of $PGL(2,p)$, as can be found in [24]. The elements of $G = PGL(2,p)$ and so also the subgroups U of prime order have at most 2 fixed points.

The fixed point free subgroups U lie in some cyclic subgroup C of order $p+1$ from a single conjugacy class and we have $N_G(U) = N_G(C)$ of order $2(p+1)$. A generator of U has $(p+1)/d$ cycles of length d if $|U| = d$. The centralizer of U then has order $d^{(p+1)/d}((p+1)/d)!$ [26] and the normalizer induces in addition an automorphism group of order $d-1$ on U. So,

$$|N_{S_{p+1}}(U)| = d^{(p+1)/d}((p+1)/d)! \cdot (d-1)$$

in this case. We have $|G:N_G(U)| = p(p-1)/2$ as the number of choices for U and as well for U'. The number of cosets $G\pi$ fixed by such subgroups U of order d is

$$\begin{aligned}\{\frac{p(p-1)}{2}\}^2|N_{S_{p+1}}(U)|\frac{1}{|G|} &= \{\frac{p(p-1)}{2}\}^2 d^{\frac{p+1}{d}}\frac{p+1}{d}!(d-1)\frac{1}{(p+1)p(p-1)} \\ &= \frac{p(p-1)}{4(p+1)}(d-1)d^{(p+1)/d}((p+1)/d)!\end{aligned}$$

If there is only one fixed point then $|U| = p$ and $|N_{S_{p+1}}(U)| = p(p-1)$.
Then we have $|G:N_G(U)| = p+1$ and

$$(p+1)^2 p(p-1)\frac{1}{(p+1)p(p-1)} = p+1$$

cosets $G\pi$ are fixed by such subgroups U.

If U of order d has two fixed points then $d|(p-1)$. We have

$$|N_{S_{p+1}}(U)| = 2d^{(p-1)/d}((p-1)/d)! \cdot (d-1).$$

U is contained in the cyclic subgroup of order $p-1$ of a dihedral subgroup D of G of order $2(p-1)$ and which lies in a single conjugacy class. Then $N_G(U) = D$ such that the number of cosets $G\pi$ fixed by such subgroups U of order d is

$$\begin{aligned}\{\frac{p(p+1)}{2}\}^2|N_{S_{p+1}}(U)|\frac{1}{|G|} &= \{\frac{p(p+1)}{2}\}^2 2d^{\frac{p-1}{d}}\frac{p-1}{d}!(d-1)\frac{1}{(p+1)p(p-1)} \\ &= \frac{p(p+1)}{4(p-1)}(d-1)d^{(p-1)/d}((p-1)/d)!\end{aligned}$$

In case of $q = 23$ we obtain that $PGL(2,23)$ has at least 4,207,092,457,345,954 long double cosets out of a total of 4,207,094,330,061,055 double cosets. So, only a very small fraction of the number of all double cosets is small. Long double cosets in $PGL(2,23)\backslash S_{24}/PSL(2,23)$ or $PSL(2,23)\backslash S_{24}/PSL(2,23)$ are obtained by splitting each long double coset in $PGL(2,23)\backslash S_{24}/PGL(2,23)$ into 2 or 4 long double cosets, respectively. Thus, also for these cases we easily obtain an even much larger number of long double cosets.

Since many isomorphism problems can be transformed into double coset problems, results on the number of long double cosets correspond to results on the number of objects with trivial automorphism group from a large scale of structures. We give an application to t-designs in the next section. For t-designs no easy way was known before to obtain bigger examples with trivial automorphism group.

A great many of instances for gluings arise from creating objects from smaller ones by adding new features or forming extensions. We use the notion of homomorphisms of group actions for a formal setting.

Definition 1 (Homomorphism of group actions) *Let G_1 be a group acting on a set Ω_1 and G_2 be a group acting on a set Ω_2. A pair $\sigma = (\sigma_\Omega, \sigma_G)$ of mappings, where σ_Ω maps Ω_1 into Ω_2 and $\sigma_G : G_1 \to G_2$ is a group homomorphism, is a homomorphism of group actions if σ is compatible with both actions, i.e. for all $g \in G_1$ and all $\omega \in \Omega_1$*

$$(\omega^g)^{\sigma_\Omega} = \omega^{\sigma_\Omega g^{\sigma_G}}.$$

If both components of σ are surjective σ is an epimorphism, if both components are bijective σ is an isomorphism.

If σ_G is not surjective orbits of the image group can be determined by the Split of Orbits Lemma from G_2-orbits. So, we futher on will restrict to the case of surjective σ_G. Then, the action of G_2 can be replaced by an appropriate action of G_1 on Ω_2. We will thus simplify the notation by these assumptions.

Theorem 5 (Homomorphism Principle) *Let a group G act on two sets Ω_1 and Ω_2 and let $\sigma : \Omega_1 \longrightarrow \Omega_2$ be compatible with both group actions. Then the preimage sets of two elements of Ω_2 from the same G-orbit intersect the same G-orbits on Ω_1. If $\sigma(\omega) = \omega'$ for two elements $\omega, \omega' \in \Omega_1$ then any $g \in G$ with $\omega^g = \omega'$ must lie in the stabilizer of $\sigma(\omega)$.*

By Theorem 5 a set of orbit representatives from the G-orbits on Ω_1 can be obtained by first determining orbit representatives from the G-orbits on Ω_2, together with their stabilizers, and then determining representatives from the stabilizer-orbits on the preimage sets of the representatives from Ω_2.

If G acts trivially on Ω_2 then the image points are *invariants*. This is a widely used method to show that two preimage points are from different orbits.

If the group acts non-trivially on Ω_2 then the stabilizers are much smaller than G. As well the preimage sets are small compared to Ω_1. So, the problem size is drastically reduced. In many cases, the stabilizers are even trivial, such that the full preimage sets can be taken as sets of representatives. Then an explicit listing can be avoided.

There are many important examples of homomorphisms, usually when there is an induced group action [32]. In computational group theory, the SOGOS system [31] made use of this. In combinatorics, multigraphs can be mapped to simple graphs setting each edge multiplicity to 1, see [12], directed graphs can be reduced to undirected graphs, labellings of edges or vertices may be omitted etc. In each case the isomorphism types of objects are just the orbits of the symmetric goup on the set of vertices acting induced on the set of objects by renaming the vertices of each object. This induced action of the symmetric group is compatible with the simplifications to simple graphs. So, these simplifications are homomorphisms of the group action. When applying the homomorphism principle, mostly no group action has to be considered when the simple graphs are extended to multigraphs or directed graphs. We only have to notice that most simple graphs have a trivial automorphism group such that the stabilizer of the object simple graph is trivial. So, in most of the cases the set of full preimages of the simplification consists of pairwise non-isomorphic objects, i. e. all multigraphs that are reduced to the same simple graph with a trivial automorpism group are pairwise non-isomorphic.

A very useful homomorphism of group actions is given by mapping each object onto its stabilizer, see also [47]. So, again the action on some set Ω is transported into an internal group action, this time the conjugation on the subgroup lattice.

Corollary 1 *If Δ is the set of objects in Ω with full stabilizer U then the orbits of $N_G(U)$ on Δ are the intersections of G-orbits with Δ. Each orbit of $N_G(U)$ on Δ has length $N_G(U):U$.*

We emphasize an important special case.

Corollary 2 *If a subgroup U is equal to its normalizer in the acting group G then all objects with stabilizer U lie in pairwise different G-orbits. In particular, if U is a maximal not normal subgroup of G then all objects fixed by U and not fixed by G lie in pairwise different G-orbits.*

Generally, an orbit of $N_G(U)$ on the set of objects with stabilizer U corresponds to that part of the G-orbit that has the same stabilizer U. Each of these orbits of $N_G(U)$ of course is in bijection to the right cosets of U.

Usually, it is much easier to determine the fixed points of a subgroup U than to find only those objects with full stabilizer U. If all minimal overgroups are known, then one can compute their fixed points as well and subtract them from the set of fixed points of U. Then there remain those with full stabilizer U. In the finite case, the number of orbits then can be determined by first computing the number of fixed points of U by the principle of exclusion-inclusion, equivalent to Möbius inversion on the subgroup lattice, and then dividing by the index of U in its normalizer. This can be done by a matrix calcul, see Burnside [15] or Kerber's book [26]. Since the action on sets of mappings is of importance we repeat the explicit formula for this case from [33].

Theorem 6 (Orbits of Mappings) *Let a group G act on a set X and let Y be another set. For any subgroup U of G the set $(Y^X)_U$ of mappings fixed by U is given by*

$$(Y^X)_U = \prod_{B\ U-orbit} \bigcup_{y \in Y} \{y\}^B$$

where the union denotes all mappings which are constant on the orbit B and the product of sets of mappings defined on disjoint sets is just the cartesian product. Then for each U a system of representatives from the orbits of $N_G(U)/U$ on

$$\prod_{B\ U-orbit} \bigcup_{y \in Y} \{y\}^B \setminus \bigcup_{U <_{max} V \leq G} \prod_{B\ V-orbit} \bigcup_{y \in Y} \{y\}^B$$

give a full system of reprentatives from the G-orbits with stabilizers from the conjugacy class of U.

The main problem in applying the Möbius inversion is the requirement that all overgroups of the subgroup U must be known. In some situations we need less information.

Definition 2 (Control of Fusion) *Let a group G act on a set Ω, let U be a subgroup of G, and let Δ be a subset of Ω. Then U controls the G-fusion on Δ if for each $\delta_1, \delta_2 \in \Delta$ and a $g \in G$ with $\delta_1^g = \delta_2$ there exists some $u \in U$ such that $\delta_1^u = \delta_2$.*

The homomorphism principle 5 describes one occurrence of such a situation. There Δ is just the preimage set of some point in Ω_2 and U is its stabilizer. Here we exhibit another case.

Theorem 7 (Localization) *Let a group G act on a set Ω, let U be a subgroup of G, and let Δ be a set of fixed points of U. If for all $\delta \in \Delta$ the stabilizer $N_G(\delta)$ controls the G-fusion on $\{U^g | g \in G,\ U^g \leq N_G(\delta)\}$ where the action is the conjugation then $N_G(U)$ controls the G-fusion on Δ.*

Thus, a control of fusion within the subgroup lattice yields a control of fusion on some exterior action of the group.

Corollary 3 *Let U be a subgroup of a group G where G is acting on a set Ω. If U is the unique subgroup of some isomorphism type in the stabilizer $N_G(\delta)$ of each point δ fixed by U then $N_G(U)$ controls the G-fusion on the set of fixed points of U.*

Another instance results from Sylow's Theorem.

Corollary 4 *Let U be a largest p-subgroup of each $N_G(\delta)$ where δ is fixed by U. Then $N_G(U)$ controls the G-fusion on the set of fixed points of U.*

If we interpret the approach as guessing the automorphism group of the objects we require that our guess at least covers a Sylow subgroup of the full automorphism group.

If U is even larger than a Sylow p-subgroup P of a stabilizer then U will have less fixed points, in general. Then the want to reduce the group that controls G-fusion on this smaller set also.

Theorem 8 (Reduction) *Let U be a subgroup of a group G where G is acting on a set Ω such that U contains a Sylow p-subgroup P of the stabilizer of each of its fixed points. Then $N_G(P) \cap N_G(U)$ controls the G fusion on the set of those fixed points that are not fixed by any proper overgroup $< U^h, U >$ of U for $h \in N_G(P)$.*

While the Moebius inversion above requires the knowledge of all minimal overgroups of U we here can construct the specific overgroups whose fixed points have to be taken out of consideration. On the remaining set of fixed points of U the smaller subgroup $D = N_G(P) \cap N_G(U)$ controls G-fusion. We remark that $DU = N_G(U)$ in this case. The objects taken out then are fixed points of a kown larger group $V =< U^h, U >$ for which we can proceed in the same way. Of course V then still contains the Sylow subgroup P of the full stabilizers such that $N_G(P)$ still controls fusion on the set of fixed points of V. So, we can apply the same technique again. But some of these groups V may lie in the same conjugacy class. So, this problem has to be solved first. If U is contained in V and V^g for some $g \in G$ then also $U^{g^{-1}}$ is contained in V. Instead of deciding the conjugacy of overgroups of U one can determine those conjugates of U that are contained in the same overgroup V. The inverses of the conjugating elements then will produce the conjugates of V containing U.

In many cases, if U is sufficiently large, each of the overgroups constructed has no fixed points. So, then again $N_G(U)$ controls the G-fusion on the set of fixed points of U. But it should be warned that even then U need not be the full stabilizer of its fixed points. This occurs for example if $U = PSL(2, 11)$ is prescribed as an automorphism group of a 5-$(12, 6, 1)$ design, a Witt design with full automorphism group M_{12}. There are two such designs which are interchanged by the normalizer $PGL(2, 11)$ of U. Prescribing $PSL(2, 23)$ as an automorphism group of a 5-$(24, 8, 1)$ design, the big Witt design, as well results in two solutions which are interchanged by $PGL(2, 23)$. The Mathieu groups, which are the full automorphism groups of these designs, are not obtained in this way.

It is a strong feature of this approach that in many cases one can decide that all objects admitting a certain group of automorphisms all must be pairwise non-isomorphic without even knowing the objects.

As an example consider $PGL(2, p)$ for some prime p. This subgroup of S_{p+1} contains a Sylow p-subgroup P of S_{p+1} and even the normalizer of P. So, all objects fixed by $PGL(2, p)$ lie in pairwise different orbits under S_{p+1}. The smallest subgroup for which this argument holds in this case is the holomorph of P acting with an additional fixed point. For any overgroup U of P any overgroup of the holomorph of P controls the S_{p+1} fusion on the set of fixed points of U. In particular $PGL(2, p)$ controls the S_{p+1} fusion on the set of all fixed points of $PSL(2, p)$.

4 Iterative Constructions

The homomorphism principle is well suited for an iteration. So, a problem is simplified in several steps and the solution strategy starts with the simplest version and stepwise tries to lift the solutions to the preimage spaces. In our aim to construct objects in each step some kind of extension occurs, depending on the actual structure.

We want to discuss some general aspects of such extensions and consider a single extension step. So, suppose an object ω and another object δ are the building parts of a new object γ, which is an extension of ω by δ. For the same pair (ω, δ) there will be several extensions, in general. These have to be classified up to isomorphism.

Usually, forming γ means to identify some structure S_1 derived from ω with a respective structure S_2 derived from δ. An automorphism of ω preserving S_1 can be applied to ω without changing the isomorphism type of the extension. The same holds for automorphisms of δ preserving S_2. So, we frequently are led to a situation where the Gluing Lemma applies. A more detailed aproach may even use prescribed stabilizers to single out objects with certain automorphisms.

The building parts will be considered as distinguished parts of the extension, and classifying these objects will only solve the isomorphism problem up to these substructures being distinguished. By selecting canonical representatives from these orbits one will obtain only *semicanonical* representatives of the general isomorphism classes.

We thus proceed in two substeps. Firstly, we classify triples (ω, δ, γ) and from these classes we, secondly, form the classes of objects γ.

Theorem 9 (Iteration Step) *Let a group A and a group B act faithfully on the space $\Omega \times \Delta \times \Gamma$ such that the projection onto $\Omega \times \Delta$ is compatible with the group action of A and the projection onto Γ is compatible with the group action of B. For each triple (ω, δ, γ) let $N_A((\omega, \delta, \gamma)) = N_B((\omega, \delta, \gamma))$. Then representatives for the B-orbits on Γ and their stabilizers in B can be obtained by the following steps.*

- *For each representative (ω, δ) from an A-orbit and its stabilizer compute representatives from the orbits of the stabilizer on the set of extensions (ω, δ, γ), where γ varies, together with their stabilizers $N_A((\omega, \delta, \gamma))$. Declare all such γ as candidates for representatives.*

- *Run through the representatives (ω, δ, γ) and do:*

  if γ is a candidate declare γ to be a representative, determine all $(\omega', \delta', \gamma)$ for this γ. For each $(\omega', \delta', \gamma)$ decide whether there exists some $b \in B$ such that $(\omega, \delta, \gamma)^b = (\omega', \delta', \gamma)$. If such a b exists then enlarge $N_A((\omega, \delta, \gamma))$ by the coset $N_A((\omega, \delta, \gamma))b$. Determine the representative $(\omega', \delta', \gamma)^a$ of its A-orbit and test whether $\gamma^a = \gamma$. If the test is negative then γ^a is removed from the set of candidates.

The condition that $N_A((\omega,\delta,\gamma)) = N_B((\omega,\delta,\gamma))$ is fulfilled if both normalizers act faithfully on the object (ω,δ,γ) and all its automorphisms are contained in A and in B.

The proof of Theorem 9 is straightforward, using the homomorphism principle twice. The two projections are the homomorphisms needed. The most interesting part is the determination of $N_B(\gamma)$. Here the bijection between an orbit and the set of right cosets of a stabilizer is used.

An important special case of Theorem 9 is the Leiterspiel by B. Schmalz[49] which computes double coset representatives in this way.

Further examples are provided by semidirect products of groups where homomorphisms from a factor group into the automorphism group of the normal subgroups have to be classified, see [29], [34], [36]. A fast graph generator relying on these principles is described in [20]. In a generator for isomers, ligands have to be placed onto places of a skeleton [48], [25], and below we will form extensions of t-design.

There are important special cases.

- Homomorphism Principle: If (ω,δ) is uniquely determined by γ then the difficult second part is not needed.

- Orderly Generation (R. Read [46], I. Faradzev [19]: If there exists a total ordering $\leq$ on $\Omega \times \Delta$ and each γ with $\gamma \leq \gamma^B$ is an extension of some (ω,δ) such that $(\omega,\delta) \leq (\omega,\delta)^A$ then only smallest elements of all orbits need to be extended and a test for $\gamma \leq \gamma^B$ suffices for these extensions.

- Canonical Generation (B.D. McKay [42]): If there exists a total ordering $\leq$ on $\Omega \times \Delta$ and a function mapping each (ω,δ) onto the minimal representative of its A-orbit and each γ contains a canonical orbit of pairs (ω,δ) then it suffices to construct only those candidates in which the extended pair (ω,δ) is canonical.

In addition, invariants may be used to reduce the computation time. Each first appearance of a new value of the invariant indicates that a new isomorphism type has been found. A comparison of such a value with the previous values can be obtained in constant time using a good hash function.

The requirements for orderly generation often can be fulfilled in combinatorial construction problems. Here, in an extension step, often some set is extended by just one element. We consider a fairly general version but explicitly fix the action.

Suppose a group G acts on a finite set X. We impose on X an ordering $<$ such that also the set 2^X of all subsets of X is lexicographically ordered. Each orbit S^G for some $S \in 2^X$ contains a lexicographically minimal element S_0 which we denote as the canonical representative with respect to $<$. In short we say $S \in canon_<(2^X, G)$ iff $S \leq S^G$. Then we have the following fundamental lemma [20].

Theorem 10 (Orderly Generation) *If* $S \in canon_<(2^X, G)$, $T \subset S$, *and* $T < S$ *then also* $T \in canon_<(2^X, G)$.

Proof: Let $S = T_1 \cup T_2$ and $T_1 < S$ but T_1 not a canonical representative. Then there exists some $g \in G$ such that $T_1^g < T_1$. If $T_1^g = \{x_1, \cdots, x_t\}$ where $x_1 < x_2 < \cdots < x_t$ then for some $i \leq t$ we have $T_1 = \{x_1, \cdots, x_{i-1}, x_i', \cdots, x_t'\}$ and $x_i < x_i'$. Since $S^g = T_1^g \cup T_2^g \supseteq \{x_1, \cdots, x_i\}$, we obtain $S^g < T_1 < S$ contradicting the hypothesis on S.

Thus, we only have to enlarge representatives T of smaller cardinality by elements x which are larger than each element in T to obtain candidates for representatives of greater cardinality. This approach can be refined by noticing that there are some further elements y larger than each element in T which can be excluded as x.

Lemma 2 (Semicanonicity) *Let* $T = \{x_1, \cdots, x_t\}$ *be canonical, where* $x_1 < x_2 < \cdots < x_t$. *Then for* $y \in x^{N_G(\{x_1,\cdots,x_i\})}$ *for* $x_i < x < x_{i+1}$ *and* $i < t$ *the set* $T \cup \{y\}$ *is not in* $canon_<(2^X, G)$. *If* $i = t$ *then if* y *is not minimal in its orbit under* $N_G(T)$ *the set* $T \cup \{y\}$ *is not in* $canon_<(2^X, G)$.

Proof: Let $y = x^g$ for some $g \in N_G(\{x_1, \cdots, x_i\})$ and some x with $x_i < x < x_{i+1}$, $i < t$. Then

$$(\{x_1, \cdots, x_i\} \cup \{x\}) < \{x_1, \cdots, x_i\}^g \cup \{x^g\} \leq T \cup \{y\}$$

such that the subset $\{x_1, \cdots, x_i, y\}$ of T is smaller than T but not canonical. Therefore by Theorem 10 also $T \cup \{y\}$ is not canonical. The second case is obvious.

The candidates obtained after removing the cases of the preceding lemma are semicanonical [44].

A test for minimality for each remaining candidate S now has to decide whether there exists some $g \in G$ such that $S^g < S$.

Often the required solutions have to fulfill some constraints. Checking these constraints is usually much faster than a canonicity check. So, a sieving with respect to the constraints will save time. One may even delay a canonicity check to the end of several extension steps hoping that after sieving only a few candidates remain. Now, if a candidate S is not minimal in its orbit then already its predecessor may not have been minimal also. In the light of Theorem 10 it is therefore useful to determine the first extension step where this non-canonicity could have been detected. Then all further extensions of this candidate must also be rejected. Depending on the selectivity of the additional constraints a delicate balance of steps with constraint checking only and steps with canonicity check combined with tracing back to the earliest detection point is needed for a fast strategy.

5 Groups and Designs

In this section, the theory shall be illustrated by a task from combinatorics. We apply the theorems of the preceding section to the problem of constructing t-(v, k, λ) designs up to isomorphism. The problems are first to find such designs for some large t but small v and then to solve the isomorphism problem for these designs. A successful strategy has been to prescribe a big group A of automorphisms and reduce the question of which k-sets should be taken as blocks to the question of which A-orbits on k-sets should be combined to form the set of blocks. So, a big group will reduce the problem size considerably.

It remains the task to find t-designs with no non-trivial automorphisms and it is to be expected that most of the designs will be of this type. But there are some parameter sets where this expectation is wrong. So, it is known that there is only one isomorphism type of designs for each of the parameter sets 2-$(7,3,1)$, 3-$(8,4,1)$, 3-$(10,4,1)$, 5-$(12,6,1)$, 5-$(24,8,1)$, each with a big automorphism group, $PGL(2,3)$, $AGL(3,4)$, $S_5^{[2]}$ M_{12}, and M_{24}, respectively. It is not clear whether there are only finitely many such cases. On the other hand, our results below will confirm that most t-designs will have a trivial automorphism group.

Firstly, for collecting k-orbits of a group one has to get these orbits. The preceding section provides at least three ways to approach this problem.

- Orderly generation
- Homomorphism principle à la *Leiterspiel (snakes and ladders)* [49]
- Prescribed stabilizers

While the use of orderly generation on a high level description is sufficiently explained in the preceding section the other two topics need some explanation. The Leiterspiel is an example for Theorem 9. It proceeds from orbits on $(k-1)$-sets to orbits on k-sets in two steps. In the first step sequences are classified consisting in the first entry of a $(k-1)$-set and in the second entry of a single point not contained in the first entry. Iterating these two up and down steps results in a growing amount of information to be stored to find out at which step representatives from previously different orbits fuse into one orbit.

The prescribed stabilizer method is useful for determining k-orbits with non-trivial stabilizers directly. So, from a knowledge of the subgroup lattice of the prescribed automorphism group one starts with a set of representatives from those conjugacy classes of subgroups that may occur as a stabilizer of a k-set. Notice that a subgroup may only fix a k-set if the sizes of point orbits may be added up to k. For illustration we explain an example from [37].

The only non-trivial subgroups of $PSL(2,23)$ that leave a 10-set invariant are subgroups of order 2. So, one can conclude by the Cauchy-Frobenius Lemma or some direct argument that there are exactly 66 orbits with stabilizers of order 2 and 290 orbits with trivial stabilizers. Since $PSL(2,23)$ is 3-homogeneous, each 10-orbit is a 3-design. The 3-designs formed by the orbits with stabilizer of order 2 have only half as many blocks as those with a trivial stabilizer. Each pair of these smaller designs then forms a design with the same number of blocks as the bigger designs. Thus, by grouping the smaller designs into pairs we get 33 designs with the same size as the remaining 290 designs. So, all 10-sets are partitioned into 323 3-designs with the same parameters. Such a partition is called a large set. Large sets are important because there some famous iterative constructions of infinite families of t-designs need large sets as a starting point.

In more general situations, a Möbius inversion as mentioned after Corollary 2 in the preceding chapter can determine those subgroups that are the full stabilizers of k-sets.

A t-design consists of a selection of k-sets as blocks such that each t-set is contained in exactly λ blocks. Constructing a t-design with a prescribed group of automorphisms A means to select appropriatwe A-orbits on k-sets. If a t-set T is contained in a blocks from an orbit K^A of k-sets then also each T^α for $\alpha in A$ is also contained in a blocks from this orbit. So, only orbit representatives need to be considered. Kramer and Mesner [27] formalized this approach by a matrix equation. The matrix contains a row for each t-orbit and a column for each k-orbit. The entry for row T^A and column K^A is the number of k-sets in K^A that contain T. Then selecting columns such that each T is contained in exactly λ k-sets from these columns amounts to solving a diophantine system of equations with a $0-1$ vector and right hand side a column vector with constant entry λ.

Though solving this problem implies solving the binary packing problem which is NP-complete, there are several algorithms which are successful at least for moderately sized problems. For very small λ one can use backtracking [42] or some clever tabu search [43, 11]. These programs constructed the largest known Steiner 5-systems on up to 244 points. That approach is also successful when there are only a few rows and many columns. For larger values of λ a version of the LLL-basis reduction algorithm is applied, see [28, 54]. The software package DISCRETA developed in Bayreuth by A. Betten, A. Wassermann and the author contains implementations of these algorithms. A graphical user interface allows an easy handling of them. The system led to many new results some of which are listed in the introduction.

The new 8-designs with automorphism group $ASL(3,3)$ were found as solutions of the Kramer-Mesner system of diophantine equations. There were many solutions and the number of isomorphism types is obtained using Theorem 1. Using DISCRETA we find that $AGL(3,3)$ is not admitted as a group of automorphisms of any of these designs. So, the normalizer $AGL(3,3)$ of $ASL(3,3)$ has orbits of length 2 on the set of these designs. Since $AGL(3,3)$ is a maximal subgroup of S_{27} and $ASL(3,3)$ is a maximal subgroup of A_{27}, see [38], then $ASL(3,3)$ is the full automorphism group of these designs. The number of isomorphism types can thus be obtained by dividing the number of solutions by 2. We have determined this number for the smaller cases in this way.

This argument was already applied by Schmalz to the classification of t-designs. The special case Corollary 2 was later used to show that all 4,996,426 7-$(33, 8, 10)$ designs with automorphism group $P\Gamma L(2, 32)$ are pairwise non-isomorphic.

The known Steiner 5-designs can also be classified by this approach. There are no 5-$(p + 1, 6, 1)$ designs admitting $PGL(2, p)$ by a result of Denniston [18]. So, all such designs found by prescribing $PSL(2, p)$ are grouped into isomorphic pairs under $PGL(2, p)$ and these are the isomorphism types. Thus, the number of isomorphism types in this case is just half the number of solutions. This could be applied to obtain the exact number of isomorphism types for $p = 11, 23, 41, 71$ and lower bounds for further incomplete sets of solutions. In particular, there are exactly 3 isomorphism types of 5-$(84, 6, 1)$ designs consisting of orbits with trivial stabilizer only and group $PSL(2, 83)$ [11].

The localization technique Theorem 8 was applied to classify the 8-$(31, 10, \lambda)$ designs with prescribed group $PSL(3, 5)$. It can also be used to solve the problems given in Kramer and Mesner's paper [27] mentioned above. There subgroups of the holomorph of C_{13} containing C_{13} had been prescribed as groups of automorphisms. For the full holomorph which is the normalizer of C_{13} in S_{13} all designs found are pairwise non isomorphic. Thus, we find 28 isomorphism types of 2-$(13, 5, 45)$ designs with this automorphism group. For the unique subgroup of index 2 there exist 890 designs allowing this group. After removing the 28 designs of the overgroup which we already considered the remaining designs fall into orbits of length 2 under the action of the holomorph which controls the S_{13} fusion. So, there result 431 new isomorphism types. Similarly, for the unique subgroup of index 3 we obtain from 24643 designs admitting this group 8205 new isomorphism types. The subgroup D_{13} admits more than 21,030,000 solutions such that with this automorphism group there exist more than 3,500,000 further isomorphism types.

We now proceed with an analysis of a well known construction and applications of it to 7- and 8-designs.

The extension method of van Leijenhorst [53] and Tran van Trung [52] builds a new design from two given designs with some appropriate parameter sets. The construction can be explained in the following way. From any t-(v, k, λ) design $\mathcal{D}$ one can obtain two smaller designs. A point x is fixed and the blocks are classified into those that contain x and those that do not contain x. Then $\{B \setminus \{x\} | B \in \mathcal{D}\}$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design, the derived design at x, and $\{B | B \in \mathcal{D}, x \notin B\}$ is a $(t-1)$-$(v-1, k, \lambda)(v-k)/(k-t+1)$ design, the residual design at x. Of course it looks promising to reverse this process. Then two given $(t-1)$-designs $\mathcal{D}_1$ with the parameters of a derived and $\mathcal{D}_2$ with the parameters of a residual design should be combined to a t-design. The construction simply has to add a new point to each block of $\mathcal{D}_1$, obtaining $\mathcal{D}_1 * \{v\}$, and then forms $\mathcal{D} = \mathcal{D}_1 * \{v\} \cup \mathcal{D}_2$. Unfortunately, only very rarely $\mathcal{D}$ is a t-design, as in the case of Alltop's Theorem [1]. But, as van Leijenhorst and Tran van Trung noticed the result is at least a $(t-1)$-$(v, k, \lambda + \lambda(v-k)/(k-t+1))$ design.

We will take a closer look at this construction. So, suppose the new point added is v. Then any $(t-1)$-set T' not containing v is contained in λ blocks from $\mathcal{D}_1 * \{v\}$ and $\lambda(v-k)/(k-t+1)$ blocks from $\mathcal{D}_2$. A $(t-1)$-set T' containing v is only contained in blocks from $\mathcal{D}_1 * \{v\}$. So, after removing v from T' and each block in $\mathcal{D}_1 * \{v\}$ we obtain the number of blocks from $\mathcal{D}_1$ containing a $(t-2)$-set. Therefore, $\mathcal{D}$ is a $(t-1)$-design if this number is equal to $\lambda + \lambda(v-k)/(k-t+1)$. But this holds because the $(t-1)$-design $\mathcal{D}_1$ is also a $(t-2)$-design with just this parameter.

We notice, that we know more about $\mathcal{D}$. Each t-set containing v is contained in exactly λ blocks. Only those t-sets not containing v may be contained in a different number of blocks.

Another important aspect results from the fact that in the construction both designs $\mathcal{D}_1$ and $\mathcal{D}_2$ can be replaced by any other design with the same parameters. So, even when starting with only two designs we can replace them by isomorphic copies to get a large number of extensions. Of course, many of them will be isomorphic but one can also obtain non-isomorphic designs in this way. We want to determine the isomorphism types in important cases. So, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are replaced by isomorphic copies then one can apply a permutation to the point set such that at least one of $\mathcal{D}_1 * \{v\}$ or $\mathcal{D}_2$ is in its original form. We therefore assume that only $\mathcal{D}_2$ is replaced by an isomorphic copy. Then we formally have applied a permutation π on the set of points $\{1, \cdots, v-1\}$ to the elements in the blocks of $\mathcal{D}_2$. We denote the result by $\mathcal{D}_2^\pi$ and get the extension $\mathcal{D}(\pi) = \mathcal{D}_1 * \{v\} \cup \mathcal{D}_2^\pi$. In this situation the Gluing Lemma 3 applies.

Theorem 11 *Let $\mathcal{D}_1$ be a $(t-1)$-$(v-1, k-1, \lambda)$ design with automorphism group A_1 and $\mathcal{D}_2$ be a $(t-1)$-$(v-1, k, \lambda(v-k)/(k-t+1))$ design with automorphism group A_2, where the point set in each case is $V' = \{1, \cdots, v-1\}$. Then there exists an isomorphism*

$$\phi : \mathcal{D}(\pi_1) \mapsto \mathcal{D}(\pi_2)$$

for permutations π_1, π_2 on V' such that ϕ fixes v if and only if

$$A_1 \pi_1 A_2 = A_1 \pi_2 A_2.$$

For the proof notice that any isomorphism ϕ fixing v has to map the derived design of $\mathcal{D}(\pi_1)$ at v onto the derived design of $\mathcal{D}(\pi_2)$ at v. The restriction to V' is an automorphism α_1 of $\mathcal{D}_1$. Similarly, the residual designs are mapped one onto the other such that $\pi_1^{-1}\phi\pi_2$ restricted to V' is an automorphism α_2 of $\mathcal{D}_2$. Thus, $\pi_1^{-1}\alpha_2\pi_2 = \alpha_1$ and $\pi_2 = \alpha_1\pi_1^{-1}\alpha_2$. On the other hand, if π_1 and π_2 lie in the same double coset modulo A_1 and A_2 then $\pi_2 = \alpha_1\pi_1^{-1}\alpha_2$, for some α_i in A_i, and $\alpha_1\pi_1^{-1}\alpha_2$ extended by the fixed point v maps $\mathcal{D}(\pi_1)$ onto $\mathcal{D}(\pi_2)$.

The group of all permutations fixing the new point v acts on the set of design extensions and its orbits are refinements of the general isomorphism classes of designs. So, we can obtain the general isomorphism classes if we can decide which of the special extension classes belong to the same general class. This can be done by the following result.

Theorem 12 *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be as before with automorphism groups A_1 and A_2 respectively. Let $\mathcal{D}(\pi_1)$ and $\mathcal{D}(\pi_2)$ be two design extensions of $\mathcal{D}_1$ and $\mathcal{D}_2$. Suppose, $\phi_1 : \mathcal{D} \mapsto \mathcal{D}(\pi_1)$ and $\phi_2 : \mathcal{D} \mapsto \mathcal{D}(\pi_2)$ are two isomorphisms from a design $\mathcal{D}$ to $\mathcal{D}(\pi_1)$ and $\mathcal{D}(\pi_2)$. Then there exists an isomorpism $\phi : \mathcal{D}(\pi_1) \mapsto \mathcal{D}(\pi_2)$ fixing some point x if and only if there exists an automorphism α of $\mathcal{D}$ mapping the point $\phi_1^{-1}(x)$ to the point $\phi_2^{-1}(x)$. In particular,*

$$Aut(\mathcal{D}(\pi_1)))_v = A_1 \cap A_2^{\pi_1}$$

The number of isomorphism types of extensions is at least

$$\frac{1}{v}|A_1 \backslash S_v / A_2|$$

For the proof notice that for any such α the composition of isomorphisms $\phi_1^{-1}\alpha\phi_2$ is an isomorphism ϕ fixing v. On the other hand, given such a ϕ we obtain α by $\alpha = \phi_1\phi\phi_2^{-1}$. The special case $\phi_1 = \phi_2$ yields the description of the stabilizer of v in the automorphism group $Aut(\mathcal{D}(\pi_1))$. From the description of the special isomorphism classes by means of double cosets in the Gluing Lemma 3 we obtain the lower bound.

More generally, if there are n_1 t-$(v-1, k-1, \lambda)$ designs with automorphism group A_1 and n_2 t-$(v-1, k, \lambda')$ designs with automorphism group A_2 and m double cosets with stabilizer order up to l then there exist at least $n_1 \times n_2 \times m/v$ isomorphism types of t-$(v, k, \lambda + \lambda')$ designs with automorphism group order up to $l \times v$.

The pairs of designs for which the extension method can be applied can be obtained from any t-design. One only has to take a derived and a residual design and then can combine them again twisted by a renaming of the points of one of the two designs. Thus, from only one t-design there results a large number of $(t-1)$-designs.

In particular each of the new 8-$(27, 13, \lambda)$ designs with automorphism group $ASL(3,3)$ first gives by Alltop's construction a 9-$(28, 14, \lambda)$ design with automorphism group $ASL(3,3)+$ and applying the extension construction with twisting to the derived and residual designs of these 9-designs using the described procedure yields $|SL(3,3)\backslash S_{27}/ASL(3,3)|/28 \geq 16913871764453533$ new 8-$(28, 14, \lambda + \lambda')$ designs with various groups of automorphisms.

We now look for situations where it can be shown that the new point v must be fixed by all isomorphisms between any extensions of two given designs. Then the double cosets above are in bijection to the isomorphism types. Also the stabilizers of v are the full automorphism groups of the designs obtained by extension and instead of the lower bound we have an exact number of isomorphism types.

Lemma 3 *Let each t-subset T of X lie in $a_1(T)$ blocks of $\mathcal{D}_1$ and in $a_2(T)$ blocks of $\mathcal{D}_2$. If for each point p there exists a T containing p such that for all permutations π $\lambda \neq a_1(T) + a_2(T^\pi)$ then in each extension the new point x is the only point such that every t-subset containing x lies in exactly λ blocks.*

Proof The t-subsets that contain x are contained in exactly the blocks that result from adding x to the blocks of $\mathcal{D}_1$. Thus those t-subsets lie in exactly λ blocks. So, any isomorphism α of any extension $\mathcal{D}_\pi$ mapping x to a point $p \neq x$ will have to map the set of blocks containing some t-subset T with $x \in T$ onto the set of blocks containing T^α where $p \in T^\alpha$. Thus, both sets of blocks must have the same cardinality. Now, T is contained in exactly λ blocks and $T' = T^\alpha$ is contained in $a_1(T')$ blocks of $\mathcal{D}_1$. The remaining blocks containing T' are from $\mathcal{D}_\pi$ such that the renaming π of the points in $\mathcal{D}_2$ causes these blocks to contain T'. So, for this π the existence of α would imply $\lambda = a_1(T') + a_2(T'^\pi)$ contrary to our assumption.

Of course it is not feasible to run through all permutations π to check whether the assumptions of the Lemma are satisfied. So, we look for sufficient conditions that are easier to check and still give the conclusion of the Lemma. First, we have orbits of the automorphism groups A_1 of $\mathcal{D}_1$ and A_2 of $\mathcal{D}_2$ on the set of all t-subsets. All T from such an orbit are contained in the same number of blocks of the respective design. A permutation π maps an orbit $T_i^{A_1}$ into several orbits $T_j^{A_2}$. Let

$$a_{ij} = |\{S : S \in T_i^{A_1}, S^\pi \in T_j^{A_2}\}|$$

where i and j run through the orbit numbers. Then

$$\sum_j a_{ij} = |T_i^{A_1}|$$

and

$$\sum_i a_{ij} = |T_j^{A_2}|.$$

If the condition $a_1(T) + a_2(T^\pi) = \lambda$ is violated then α cannot exist. Therefore all a_{ij} where $a_1(T_i^{A_1}) + a_2(a_j^{A_2}) \neq \lambda$ are zero. If the remaining system of Diophantine equations has no solutions then α cannot exist. So, this set of equations yields a sufficient condition to conclude that all isomorphism types of extensions of two particular designs are in bijection to the double cosets of A_1 and A_2 in S_{v-1}.

A special situation occurs when a prescribed automorphism group is transitive on the set of points.

A very prominent example is formed by the smallest 6-designs. These designs have parameters 6-$(14, 7, 4)$ and are constructed by Alltop's Theorem from a 5-$(13, 6, 4)$ design, [28]. The automorphism group of the 5-design is C_{13} and there exist exactly 24 solutions of the Kramer-Mesner system of equations. Thus the isomorphism types are given by the orbits of the normalizer $Hol(C_{13})$ of C_{13} in S_{13} on the set of points which have sizes 1 and 12. So, there are exactly 2 isomorphism types of 5-$(13, 6, 4)$ designs with automorphism group C_{13}. By an argument of Kreher and Radziszowski in [28] the isomorphism types of the extensions often can also be determined in such a situation.

In Alltop's construction, the blocks of the residual design that by which the derived design is extended are uniquely determined by the derived design. So, all automorphisms of $\mathcal{D}$ extend to the extended design $\mathcal{D}^+$. Therefore,

$$Aut(\mathcal{D}) \;=\; Aut(\mathcal{D}^+)_v.$$

Taking the derived designs at other points thus give designs whose automorphism groups are the corresponding other point stabilizers.

Theorem 13 *Let A be the full automorphism group of t-(v, k, λ) designs where $v = 2k+1$ and t is even or $\lambda = \frac{1}{2}\binom{v-t}{k-t}$. If A acts transitively on the point set but has no transitive extension then two Alltop extensions $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ of t-(v, k, λ) designs $\mathcal{D}_1$ and $\mathcal{D}_2$ with full automorphism group A are isomorphic if and only if $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic.*

The proof is immediate from the fact that the full automorphism group of an Alltop extension here either is transitive or has the new point as a fixed point. So, if A cannot be transitively extended then the derived design at the new point is not isomorphic to any other derived design and thus characterizes the isomorphism type of the Alltop extension.

In case of the 5-$(13, 6, 4)$ design the extended design still has C_{13} as its full automorphism group with an additional fixed point 14. So, all other points form just one orbit and have a trivial stabilizer. Therefore the other derived designs have trivial automorphism groups. In particular, different isomorphism types of 5-$(13, 6, 4)$ designs with automorphism group C_{13} extend to different isomorphism types of 6-$(14, 7, 4)$ designs. The new 9-$(28, 14, \lambda)$ designs are obtained from 8-$(27, 13, \lambda)$ designs by using Alltop's construction. Here, the automorphism group $ASL(3, 3)$ acts transitively but cannot be transitively extended. Otherwise, the extended group would have to be at least 2-transitive and there is even no primitive group on 28 points different from the alternating and the full symmetric group containing $ASL(3, 3)$, see for example [16]. So, different isomorphism types of 8-$(27, 13, \lambda)$ designs with automorphism group $ASL(3, 3)$ extend to different isomorphism types of 9-$(28, 14, \lambda)$ designs.

There are many further situations where the automorphism group is transitive and Alltop's construction applies. So, it is sufficient in these cases to verify that the group is not the stabilizer of a point in a primitive group. Then the Theorem allows to determine the isomorphism types of Alltop extensions from the isomorphism types of the given designs.

We now again consider the general situation of extensions and assume a transitive automorphism group on the design with the larger block size.

Theorem 14 *Let A_1 be the automorphism group of a $(t-1)$-$(v-1, k-1, \lambda)$ design $\mathcal{D}_1$ and A_2 the automorphism group of a $(t-1)$-$(v-1, k, \lambda(v-k)/(k-t+1))$ design $\mathcal{D}_2$ both defined on a point set V. Let A_2 act transitively on the set of $v-1$ points and let none of the extensions be a t-(v, k, λ) design. Then the isomorphism types of extensions of $\mathcal{D}_1$ and $\mathcal{D}_2$ to a $(t-1)$-$(v, k, \lambda(v-t+1)/(k-t+1))$ design are in bijection to the double cosets $A_1 \backslash Sym(V)/A_2$. The automorphism group of an extension of $\mathcal{D}_1$ by $\mathcal{D}_2^{\pi}$ for some permutation π is $A_1 \cap {A_2}^{\pi}$.*

Proof By the Lemma it suffices to show that for each point of the point set different from the added point there exists a t-subset T such that this T is not contained in λ blocks of the extension design. Since A_2 is transitive on these points, each orbit of A_2 on t subsets contains a block containing a fixed point p. So, we only have to find one orbit T^{A_2} such that $a_1(T) + a_2(T)$ is different from λ. This means that the extension is not a t-design, as assumed in the Theorem.

Corollary 5 *If for a prescribed automorphism group A there exist n_1 designs with parameter set $(t-1)$-$(v-1, k-1, \lambda)$ and n_2 designs with parameter set $(t-1)$-$(v, k, \lambda(v-k)/(k-t+1))$ then under the assumptions of the last Theorem there exist $n_1 \cdot n_2 \cdot |A \backslash Sym(V)/A|$ isomorphism types of extensions.*

In the last Theorem, it suffices to find only one t-orbit of A_2 such that any t-subset T from this orbit lies in strictly more than λ blocks of $\mathcal{D}_2$. Then one can also conclude that none of the extensions will be a t-design. This holds for example for each of the 113 7-$(24, 9, 40)$ designs with automorphism group $PGL(2, 23)$ as can be verified by DISCRETA. So, by the Corollary forming the extensions with any of the 138 7-$(24, 8, 5)$ designs with automorphism group $PSL(2, 23)$ yields as many isomorphism types of 7-$(25, 9, 45)$ designs as there are double cosets of these automorphism groups in S_{24}. Thus, we obtain in this way exactly

$$113 \times 138 \times 8,414,188,491,217,916 = 131,210,855,332,052,182,104$$

isomorphism types of 7-$(25, 9, 45)$ designs.

In the introduction we have given a table on extensions of designs with results obtained from this approach. We discuss some entries of that table.

- The lower bound for the number of 7-$(26, 10, 342)$ designs in the last row is obtained by multiplying the numbers of designs with the parameters 7-$(25, 9, 54)$ and 7-$(25, 10, 288)$ in that row by the number of double cosets of the identity in S_{25}, i. e. 25!, and then dividing by 26, which is the maximal number of designs that may be isomorphic after making the new point an ordinary point. It is likely, that all these designs are pairwise non-isomorphic such that the last division is superfluous.

- The new point v is distinguished in some extensions in the following cases:

  Row No. 1: the third and fourth 7-$(24, 10, 240)$ designs from the list in [4], Row No. 3: all extensions, Row No. 4: all of 10 extensions tested, Row No. 5: all of 15 extensions tested, Row No. 6: each of 21 7-$(26, 9, 54)$ designs, Row No. 8: at least five 7-$(27, 11, 2295)$ designs, Row No. 9: the nineth 7-$(27, 11, 2295)$ design from the list in [4], Row No.10: all of 500 extensions tested.

- Row number 8 shows an example of totally different automorphism groups.

- Row number 7 is interesting, because the 7-$(26, 13, 13524)$ design results from first taking the residual design of a 7-$(26, 12, 5796)$ design and then extending that by Alltop's construction. Thus, here the existence of only one design suffices for the Tran van Trung-van Leijenhorst construction. Taking the residual design reduces the automorphism group to the stabilizer of a point, in this case $A\Gamma L(1, 25)$.

- Row 11 with the group $Sp(6, 2)$ acting on 28 points is similar. There are some further values of λ for which there exists a 7-$(26, 13, \lambda)$ design with automorphism group $Sp(6, 2)$. For each of them the same construction can be applied. In each of these cases we cannot give the number of isomorphism types of extensions.

- Row 12 uses the results from row 1 and row 4. So, in this case we are supposed to see all kinds of subgroups of $PGL(2, 23)$ as automorphism groups. We then can combine any such pair and form their double cosets in S_{25}. This number of double cosets then has to be multiplied with the number of solutions belonging to these automorphism groups. So, this illustrates the theory given above.

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