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# Simple 8-Designs with Small Parameters

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**Abstract.** We show the existence of simple 8-(31,10,93) and 8-(31,10,100) designs. For each value of $\lambda$ we show 3 designs in full detail. The designs are constructed with a prescribed group of automorphisms $\mathrm{PSL}(3,5)$ using the method of KRAMER and MESNER [8]. They are the first 8-designs with small parameters which are known explicitly. We do not yet know if $\mathrm{PSL}(3,5)$ is the *full* group of automorphisms of the given designs. There are altogether 138 designs with $\lambda = 93$ and 1658 designs with $\lambda = 100$ and $\mathrm{PSL}(3,5)$ as *a group of automorphisms*. We prove that they are all pairwise non-isomorphic. For this purpose, a brief account on the intersection numbers of these designs is given. The proof is done in two different ways. At first, a quite general group theoretic observation shows that there are no isomorphisms. In a second approach we use the block intersection types as invariants, they classify the designs completely.

**Keywords:** $t$-design, Kramer-Mesner method, intersection number, isomorphism problem, group action.

## 1. Introduction

In this paper, $t$-designs with prescribed automorphism group are constructed. The method was introduced by KRAMER and MESNER in [8]. We choose as group $\mathrm{PSL}(3,5)$ and construct 8-$(31,10,\lambda)$ designs with two different values of $\lambda$. We get 1658 designs with $\lambda = 100$ and 138 designs with $\lambda = 93$. Some questions immediately arise:

1. Are the designs all distinct, i.e. pairwise non-isomorphic, or, if not, which of them form a transversal of the isomorphism classes?

2. What is the full group of automorphisms of each of the designs?

3. Are there designs for other values of $\lambda$?

4. Are there more designs with a possibly smaller group of automorphisms?

In the following sections, we will answer question 1 twice and question 3 partly. Problem 2 would be easily solved if it were known that $\mathrm{PSL}(3,5)$ is a maximal subgroup of $S_{31}$. Note that this fact would imply that 1 is true. Indeed, we will show in section 7 that designs with the same automorphism group are isomorphic if and only if they are isomorphic under the normalizer of this group.

The plan of this paper is the following: In sections 2 and 3, the method of Kramer and Mesner is briefly sketched. We will give a list of all orbits of the group on 10-subsets which is needed to describe the designs. In section 4, we recall basic facts about parameters of designs and about intersection numbers. We introduce the equations of MENDELSOHN and KÖHLER. Moreover, we define intersection numbers of higher order and list the relevant generalizations of the parameter equations. These are due to TRAN VAN TRUNG, QUI-RONG WU and DALE M. MESNER. We also define global intersection numbers and use the generalized equations to provide means for checking them.

The following two sections 5 and 6 are devoted to the 8-$(31,10,100)$ and 8-$(31,10,93)$ designs, respectively. For each of these cases, the parameter equations are shown. As the numbers involved tend to become quite large in some cases, this can be of great help avoiding tedious hand-calculations. In fact, all these calculations were done by a computer using a long-integer arithmetic. For each value of $\lambda$, 3 designs are listed in full detail. They should serve as examples. The interested reader may reconstruct the full set of designs using our program DISCRETA [6] which is freely available on the internet. The numbering of designs is imposed by the order in which the solutions are computed by the equation solver of DISCRETA (this program is deterministic, so that the order is always the same). See [17] for a more detailed treatment on solving large equation systems with integral coefficients.

Finally, in section 7 the two announced proofs of problem 1 are given. The first applies group theoretic methods together with some (small) computer calculations. The second is a more combinatorial one. It uses intersection numbers as invariants to show that no two designs are isomorphic.

Problem 4 is beyond the scope of this paper.

## 2. The Group and its Orbits

We denote the elements of the field $GF(5)$ by $0,1,2,3,4$. The elements of the projective geometry $PG_2(5)$ can be identified with the one-dimensional subspaces of $GF(5)^3$. We number them in the following way using representatives $(a,b,c)^t$:

| | | | | |
|---|---|---|---|---|
| $1 \widehat{=} (1,0,0)^t$ | $8 \widehat{=} (1,0,1)^t$ | $15 \widehat{=} (3,1,1)^t$ | $22 \widehat{=} (0,3,1)^t$ | $29 \widehat{=} (2,4,1)^t$ |
| $2 \widehat{=} (0,1,0)^t$ | $9 \widehat{=} (2,0,1)^t$ | $16 \widehat{=} (4,1,1)^t$ | $23 \widehat{=} (1,3,1)^t$ | $30 \widehat{=} (3,4,1)^t$ |
| $3 \widehat{=} (1,1,0)^t$ | $10 \widehat{=} (3,0,1)^t$ | $17 \widehat{=} (0,2,1)^t$ | $24 \widehat{=} (2,3,1)^t$ | $31 \widehat{=} (4,4,1)^t$ |
| $4 \widehat{=} (2,1,0)^t$ | $11 \widehat{=} (4,0,1)^t$ | $18 \widehat{=} (1,2,1)^t$ | $25 \widehat{=} (3,3,1)^t$ | |
| $5 \widehat{=} (3,1,0)^t$ | $12 \widehat{=} (0,1,1)^t$ | $19 \widehat{=} (2,2,1)^t$ | $26 \widehat{=} (4,3,1)^t$ | |
| $6 \widehat{=} (4,1,0)^t$ | $13 \widehat{=} (1,1,1)^t$ | $20 \widehat{=} (3,2,1)^t$ | $27 \widehat{=} (0,4,1)^t$ | |
| $7 \widehat{=} (0,0,1)^t$ | $14 \widehat{=} (2,1,1)^t$ | $21 \widehat{=} (4,2,1)^t$ | $28 \widehat{=} (1,4,1)^t$ | |

The group $\mathrm{PSL}(3,5)$, represented as a permutation group on $PG_2(5)$ is generated by the following permutations:
(1 2 6)(3 4 5)(8 12 16 11 27 28)(9 17 20 10 22 24)(13 21 15 31 23 29)(14 26 19 30 18 25),
(1 3 5 4 6)(8 13 18 23 28)(9 19 29 14 24)(10 25 15 30 20)(11 31 26 21 16),
(1 4 5 6)(8 21 18 16)(9 30 29 20)(10 14 15 24)(11 23 26 28)(12 17 27 22)(13 31)(19 25),

(1 7 27 6)(2 11)(3 9 22 15)(4 10 12 19)(5 8 17 23)(13 25 18 14)(16 31)(20 30 24 29).
The group is of order

$$\frac{(5^3-1)(5^3-5)(5^3-5^2)}{5-1} = (5^2+5+1)(5^2-1)(5-1)5^3 = 372000. \qquad (1)$$

We are now going to construct $t$-$(v,k,\lambda)$ designs on the set $V = PG_2(5)$ of vertices and with $A = \mathrm{PSL}(3,5)$ contained in their automorphism groups. So, the parameter $v = 31$, and we are after 8-designs, so $t = 8$. Moreover we put $k = 10$ and leave $\lambda$ open, in fact our method of construction showed that $\lambda = 93$ and $\lambda = 100$ are fine.

We start with computing orbits of $A$ on $i$-sets, $i \leq 10$. The numbers of orbits are shown in table 1.

*Table 1.* Number of orbits of $\mathrm{PSL}(3,5)$ on $i$-subsets of $PG_2(5)$

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| # $i$-orbits of $A$ | 1 | 1 | 1 | 2 | 3 | 5 | 12 | 22 | 42 | 92 | 174 |

A design with parameter $k = 10$ is a collection of blocks of size 10. If it has $A$ as subgroup of its automorphism group, the set $\mathcal{B}$ of blocks decomposes into a collection of orbits of $A$ on 10-sets. In order to describe the design, we only need to know which orbits (among the total set of orbits of $A$ on the set $\binom{V}{10}$ of *all* the 10-subsets of the set $V$ of points) belong to the design. Therefore, we label the $A$-orbits and refer to these numbers later on.

The following table shows all 10-orbits of $A$ on $V$. The stabilizer order is indicated by a subscript. The orbit length is the index of the stabilizer in $A$. We give the lexicographically minimal representative within each orbit. This list of representatives is not lexicographically ordered, due to the fact that we do not generate orbits via orderly generation. Instead of orderly generation, we use an algorithm *Leiterspiel* [12] (snakes and ladders) to provide orbit representatives and further knowledge needed for the evaluation of Kramer-Mesner matrices (see below). As the representatives all start with the sequence $1, 2, 3, 4, \ldots$ of consecutive numbers, only the last of these numbers is shown. The beginning is replaced by the symbol "$\ldots$".

**10-orbits:**
1: $\{\ldots, 5, 7, 16, 20, 24, 28\}_{800}$
2: $\{\ldots, 10\}_{80}$
3: $\{\ldots, 5, 7, 8, 12, 21, 25\}_{4}$
4: $\{\ldots, 9, 12\}_{2}$
5: $\{\ldots, 5, 7, 8, 12, 16, 20\}_{2}$
6: $\{\ldots, 5, 7, 8, 12, 16, 24\}_{2}$
7: $\{\ldots, 5, 7, 8, 12, 14, 25\}_{1}$
8: $\{\ldots, 5, 7, 8, 9, 13, 17\}_{1}$
9: $\{\ldots, 5, 7, 8, 9, 13, 30\}_{2}$
10: $\{\ldots, 5, 7, 8, 12, 13, 21\}_{2}$
11: $\{\ldots, 5, 7, 8, 9, 12, 21\}_{2}$
12: $\{\ldots, 5, 7, 8, 9, 12, 29\}_{4}$
13: $\{\ldots, 8, 12, 13\}_{8}$
14: $\{\ldots, 8, 12, 14\}_{4}$
15: $\{\ldots, 8, 12, 19\}_{6}$
16: $\{\ldots, 5, 7, 8, 9, 12, 13\}_{1}$
17: $\{\ldots, 5, 7, 8, 9, 12, 28\}_{1}$
18: $\{\ldots, 5, 7, 8, 12, 14, 18\}_{2}$
19: $\{\ldots, 5, 7, 8, 12, 14, 28\}_{2}$
20: $\{\ldots, 5, 7, 8, 9, 13, 16\}_{2}$
21: $\{\ldots, 5, 7, 8, 9, 13, 28\}_{4}$
22: $\{\ldots, 5, 7, 8, 12, 13, 20\}_{1}$
23: $\{\ldots, 5, 7, 8, 9, 12, 24\}_{1}$
24: $\{\ldots, 5, 7, 8, 9, 12, 17\}_{2}$
25: $\{\ldots, 5, 7, 8, 12, 14, 24\}_{4}$
26: $\{\ldots, 5, 7, 8, 9, 13, 20\}_{1}$
27: $\{\ldots, 5, 7, 8, 9, 14, 18\}_{1}$
28: $\{\ldots, 5, 7, 8, 9, 13, 15\}_{1}$
29: $\{\ldots, 5, 7, 8, 12, 14, 19\}_{1}$
30: $\{\ldots, 5, 7, 8, 9, 12, 22\}_{1}$
31: $\{\ldots, 5, 7, 8, 9, 12, 14\}_{1}$
32: $\{\ldots, 5, 7, 8, 9, 12, 18\}_{1}$
33: $\{\ldots, 5, 7, 8, 9, 13, 22\}_{1}$
34: $\{\ldots, 5, 7, 8, 9, 13, 14\}_{1}$
35: $\{\ldots, 5, 7, 8, 9, 13, 27\}_{1}$
36: $\{\ldots, 5, 7, 8, 9, 14, 31\}_{2}$
37: $\{\ldots, 5, 7, 8, 9, 12, 27\}_{1}$
38: $\{\ldots, 5, 7, 8, 9, 13, 23\}_{1}$

39: $\{\dots,5,7,8,9,10,12\}_{2}$
40: $\{\dots,5,7,8,9,10,15\}_{8}$
41: $\{\dots,5,7,8,9,14,19\}_{2}$
42: $\{\dots,5,7,8,9,14,27\}_{4}$
43: $\{\dots,5,7,8,9,14,15\}_{2}$
44: $\{\dots,5,7,8,13,15,22\}_{20}$
45: $\{\dots,4,7,8,12,16,20,28\}_{2}$
46: $\{\dots,4,7,8,12,16,20,24\}_{4}$
47: $\{\dots,4,7,8,12,16,21,25\}_{2}$
48: $\{\dots,4,7,8,12,14,25,29\}_{1}$
49: $\{\dots,4,7,8,9,12,19,26\}_{1}$
50: $\{\dots,4,7,8,9,12,19,28\}_{1}$
51: $\{\dots,4,7,8,12,13,21,29\}_{2}$
52: $\{\dots,4,7,8,9,12,18,24\}_{2}$
53: $\{\dots,4,7,8,9,12,18,30\}_{2}$
54: $\{\dots,4,7,8,12,15,18,26\}_{8}$
55: $\{\dots,4,7,8,12,21,25,26\}_{16}$
56: $\{\dots,4,7,8,12,16,20,29\}_{4}$
57: $\{\dots,4,7,8,12,16,20,25\}_{1}$
58: $\{\dots,4,7,8,12,16,20,21\}_{2}$
59: $\{\dots,4,7,8,12,15,19,29\}_{1}$
60: $\{\dots,4,7,8,9,13,20,26\}_{2}$
61: $\{\dots,4,7,8,9,12,20,29\}_{1}$
62: $\{\dots,4,7,8,9,13,20,21\}_{1}$
63: $\{\dots,4,7,8,12,15,19,23\}_{1}$
64: $\{\dots,4,7,8,9,12,16,20\}_{2}$
65: $\{\dots,4,7,8,9,12,13,29\}_{1}$
66: $\{\dots,4,7,8,12,16,19,29\}_{1}$
67: $\{\dots,4,7,8,9,12,19,23\}_{2}$
68: $\{\dots,4,7,8,9,12,16,17\}_{1}$
69: $\{\dots,4,7,8,9,12,13,21\}_{1}$
70: $\{\dots,4,7,8,9,12,16,29\}_{2}$
71: $\{\dots,4,7,8,12,16,19,20\}_{2}$
72: $\{\dots,4,7,8,12,15,23,29\}_{2}$
73: $\{\dots,4,7,8,9,13,20,24\}_{1}$
74: $\{\dots,4,7,8,9,12,20,26\}_{6}$
75: $\{\dots,4,7,8,9,12,17,20\}_{1}$
76: $\{\dots,4,7,8,12,19,20,23\}_{2}$
77: $\{\dots,4,7,8,9,12,18,29\}_{1}$
78: $\{\dots,4,7,8,12,16,19,23\}_{2}$
79: $\{\dots,4,7,8,12,16,21,28\}_{2}$
80: $\{\dots,4,7,8,9,12,18,26\}_{1}$
81: $\{\dots,4,7,8,9,12,18,20\}_{1}$
82: $\{\dots,4,7,8,9,13,17,20\}_{1}$
83: $\{\dots,4,7,8,9,12,18,31\}_{1}$
84: $\{\dots,4,7,8,9,12,20,24\}_{1}$
85: $\{\dots,4,7,8,12,13,20,24\}_{1}$
86: $\{\dots,4,7,8,9,12,19,31\}_{1}$
87: $\{\dots,4,7,8,9,12,13,20\}_{1}$
88: $\{\dots,4,7,8,9,12,16,19\}_{1}$
89: $\{\dots,4,7,8,12,13,21,24\}_{1}$
90: $\{\dots,4,7,8,9,12,13,18\}_{1}$
91: $\{\dots,4,7,8,12,13,21,26\}_{1}$
92: $\{\dots,4,7,8,9,12,21,24\}_{2}$
93: $\{\dots,4,7,8,9,12,15,29\}_{1}$
94: $\{\dots,4,7,8,9,13,17,28\}_{1}$
95: $\{\dots,4,7,8,9,12,15,20\}_{1}$
96: $\{\dots,4,7,8,12,14,28,29\}_{1}$
97: $\{\dots,4,7,8,9,12,19,20\}_{1}$
98: $\{\dots,4,7,8,12,14,18,29\}_{1}$
99: $\{\dots,4,7,8,9,17,20,25\}_{1}$
100: $\{\dots,4,7,8,9,12,16,28\}_{2}$
101: $\{\dots,4,7,8,9,12,13,17\}_{1}$
102: $\{\dots,4,7,8,9,17,19,23\}_{1}$
103: $\{\dots,4,7,8,9,12,15,27\}_{2}$
104: $\{\dots,4,7,8,9,12,18,19\}_{1}$
105: $\{\dots,4,7,8,9,17,18,26\}_{2}$
106: $\{\dots,4,7,8,9,12,17,26\}_{3}$
107: $\{\dots,4,7,8,9,17,19,26\}_{1}$
108: $\{\dots,4,7,8,9,12,15,28\}_{1}$
109: $\{\dots,4,7,8,9,12,15,21\}_{1}$
110: $\{\dots,4,7,8,9,12,19,24\}_{1}$
111: $\{\dots,4,7,8,9,12,18,27\}_{1}$
112: $\{\dots,4,7,8,9,17,20,23\}_{1}$
113: $\{\dots,4,7,8,9,12,15,17\}_{1}$
114: $\{\dots,4,7,8,9,12,20,21\}_{1}$
115: $\{\dots,4,7,8,9,12,20,28\}_{1}$
116: $\{\dots,4,7,8,9,12,14,20\}_{1}$
117: $\{\dots,4,7,8,9,13,21,27\}_{1}$
118: $\{\dots,4,7,8,9,12,28,31\}_{2}$
119: $\{\dots,4,7,8,9,12,19,21\}_{1}$
120: $\{\dots,4,7,8,9,12,14,19\}_{1}$
121: $\{\dots,4,7,8,9,12,21,31\}_{1}$
122: $\{\dots,4,7,8,9,12,15,18\}_{1}$
123: $\{\dots,4,7,8,9,13,14,17\}_{1}$
124: $\{\dots,4,7,8,9,12,13,19\}_{1}$
125: $\{\dots,4,7,8,9,14,17,25\}_{2}$
126: $\{\dots,4,7,8,12,14,25,28\}_{1}$
127: $\{\dots,4,7,8,9,12,14,28\}_{1}$
128: $\{\dots,4,7,8,9,13,21,24\}_{2}$
129: $\{\dots,4,7,8,9,12,17,18\}_{1}$
130: $\{\dots,4,7,8,9,12,18,25\}_{2}$
131: $\{\dots,4,7,8,9,12,27,28\}_{1}$
132: $\{\dots,4,7,8,9,12,13,28\}_{1}$
133: $\{\dots,4,7,8,9,12,15,19\}_{1}$
134: $\{\dots,4,7,8,9,12,14,18\}_{1}$
135: $\{\dots,4,7,8,9,12,19,27\}_{1}$
136: $\{\dots,4,7,8,9,12,18,21\}_{1}$
137: $\{\dots,4,7,8,12,13,20,21\}_{1}$
138: $\{\dots,4,7,8,12,20,21,28\}_{4}$
139: $\{\dots,4,7,8,9,13,17,21\}_{1}$
140: $\{\dots,4,7,8,12,14,18,25\}_{1}$
141: $\{\dots,4,7,8,9,13,18,20\}_{2}$
142: $\{\dots,4,7,8,9,13,14,20\}_{1}$
143: $\{\dots,4,7,8,9,13,15,20\}_{2}$
144: $\{\dots,4,7,8,12,14,24,25\}_{2}$
145: $\{\dots,4,7,8,9,12,14,21\}_{1}$
146: $\{\dots,4,7,8,12,15,18,21\}_{2}$
147: $\{\dots,4,7,8,9,13,14,18\}_{1}$
148: $\{\dots,4,7,8,9,13,15,17\}_{2}$
149: $\{\dots,4,7,8,12,13,24,26\}_{4}$
150: $\{\dots,4,7,8,9,14,17,18\}_{2}$
151: $\{\dots,4,7,8,9,12,13,27\}_{2}$
152: $\{\dots,4,7,8,12,15,19,20\}_{4}$
153: $\{\dots,4,7,8,9,13,15,27\}_{4}$
154: $\{\dots,4,7,8,12,13,20,30\}_{2}$
155: $\{\dots,4,7,8,9,12,13,15\}_{2}$
156: $\{\dots,4,7,8,9,13,15,28\}_{2}$
157: $\{\dots,4,7,8,9,12,14,16\}_{4}$
158: $\{\dots,4,7,8,9,12,16,27\}_{8}$
159: $\{\dots,4,7,8,9,12,13,16\}_{6}$
160: $\{\dots,4,7,8,9,12,13,14\}_{6}$
161: $\{\dots,4,7,8,9,13,14,27\}_{6}$
162: $\{\dots,4,7,8,9,14,17,21\}_{3}$
163: $\{\dots,4,7,8,9,12,14,27\}_{2}$
164: $\{\dots,4,7,8,12,14,18,19\}_{4}$
165: $\{\dots,4,7,8,9,12,14,15\}_{2}$
166: $\{\dots,4,7,8,9,13,18,25\}_{20}$
167: $\{\dots,4,7,8,9,13,14,15\}_{12}$
168: $\{\dots,3,7,8,12,15,19,20,23\}_{2}$
169: $\{\dots,3,7,8,12,13,20,21,24\}_{3}$
170: $\{\dots,3,7,8,12,14,21,24,28\}_{2}$
171: $\{\dots,3,7,8,12,14,21,28,31\}_{4}$
172: $\{\dots,3,7,8,12,15,19,21,26\}_{24}$
173: $\{\dots,3,7,8,12,14,18,21,31\}_{3}$
174: $\{\dots,3,7,8,13,21,24,29,30\}_{120}$

## 3. Orbit Selection

The designs are constructed using the Kramer-Mesner matrix $M_{t,k}^{A} = (m_{ij})$ which consists in our case of 42 rows and 174 columns (recall that $A = \mathrm{PSL}(3,5)$, $t = 8$ and $k = 10$, cf. table 1). The entry $m_{ij}$ is the number of $k$-subsets in the $j$-th orbit of $A$ on $k$-subsets containing the representative of the $i$-th orbit on $t$-subsets. Hence, the $\{0,1\}$-solutions of the diophantine system of equations

$$M_{t,k}^{A}\, x^{\mathrm{t}} \;=\; (\lambda,\dots,\lambda)^{\mathrm{t}} \text{ with } x = (x_1, x_2, \dots, x_l) \text{ and } x_j \in \{0,1\} \text{ for } 1 \le j \le l \tag{2}$$

are exactly the possible ways of choosing suitable block orbits (the chosen columns) which fulfil all the conditions of a $t$-$(v,k,\lambda)$ design admitting the prescribed automorphism group $A$. Namely, such a solution is a collection of group orbits on $k$-sets such that each representative $T$ of a $t$-orbit is contained in exactly $\lambda$ $k$-sets from all the chosen $k$-orbits.

This system was completely solved by the LLL-based algorithm as described in [17]. There are exactly 138 solutions for $\lambda = 93$ and 1658 solutions for $\lambda = 100$. No solutions exist for other values of $\lambda \leq 126$ for this system of equations.

The enumeration of all solutions is a backtracking-algorithm over the integral linear combinations of the LLL-reduced basis-vectors of the corresponding Kramer-Mesner system. To speed up the search one can parallelize the program. [3] describes a parallel version of the algorithm. Nevertheless, the designs here were found with the sequential version of the program within a few hours.

## 4. Intersection Numbers of Designs

In this section, we recall some basic facts about parameters and intersection numbers of designs. We will make use of intersection numbers in section 7 when proving the fact that the 8-designs with $\mathrm{PSL}(3,5)$ are pairwise non-isomorphic. Intersection numbers have a long history in design theory, early results were obtained by Mendelsohn [10] and Stanton and Sprott [14]. They can be generalized to higher $t$, we will show them soon. The equations of Köhler [7] support the evaluation of these formulae. We will also speak about generalized intersection numbers, which already appeared in [10]. Recent progress was made by Tran van Trung, Qiu-Rong Wu and Dale M. Mesner [16].

Let $\mathcal{D} = (V, \mathcal{B})$ be a $t$-$(v, k, \lambda)$ design on the set of points $V$ with $|V| = v$. Let $\mathcal{B} = \{B_1, \ldots, B_b\}$ be the blocks with $B_i \subseteq V$ for $i = 1, \ldots, b$.

Fix disjoint subsets $I$ and $J$ of $V$ with $|I| = i$, $|J| = j$ and $i + j \leq t$. Define

$$\lambda_{i,j} = |\{B \in \mathcal{B} |\, I \subseteq B \wedge J \cap B = \emptyset\}| \tag{3}$$

($\lambda_{t,0} = \lambda$, $\lambda_{1,0} = r$ the number of blocks which contain a given point and $\lambda_{0,0} = b$). Ray-Chaudhuri and Wilson proved in [11] that these numbers $\lambda_{i,j}$ are independent of the choice of the sets $I$ and $J$ (depending only on their cardinalities $i$ and $j$). They can be computed by the following formula

$$\lambda_{i,j} \;=\; \lambda \binom{v-i-j}{k-i} \Big/ \binom{v-t}{k-t}. \tag{4}$$

The following recursion holds for $i + j \leq t$:

$$\lambda_{i,j} = \lambda_{i,j-1} - \lambda_{i+1,j-1}. \tag{5}$$

This is the same recursion as in the well-known Pascal-triangle of binomial coefficients. Here, one also speaks of the *intersection triangle* of the design. For sake of simplicity, set $\lambda_i := \lambda_{i,0}$.

For an arbitrary fixed $m$-subset $M$ of $V$ define for $0 \leq i \leq m$:

$$\alpha_i(M) := |\{B \in \mathcal{B} |\, |M \cap B| = i\}| \tag{6}$$

the *i-th intersection number* of $M$ with $\mathcal{D}$. The reference to the set $M$ will sometimes be omitted. It should then be clear from the context which set $M$ we are referring to.

Let $M$ be an arbitrary $m$-subset of $V$. Fix $j$ with $0 \leq j \leq t$. Counting the set

$$\{ (J, B) \mid |J| = j, B \in \mathcal{B}, J \subseteq B \cap M \}$$

in two different ways, one arrives at the equations of MENDELSOHN [10]:

$$\sum_{i=j}^{k} \binom{i}{j} \alpha_i(M) = \binom{m}{j} \lambda_{j,0} \tag{7}$$

(for all $j = 0, 1, \ldots, t$). Writing down the system of equations we obtain a matrix which is nearly upper triangular. Depending on the relation between $m$ and $t$ one either has an upper triangular matrix (for $m \leq t$) or a matrix which is upper triangular in its left $(t+1) \times (t+1)$ submatrix but which has some additional columns $t+2, \ldots, m+1$ (for $m > t$). Most important for us are the cases when $m = k$ and $M = B_0 \in \mathcal{B}$ is chosen to be a block of the design itself. In this case, $\alpha_k(B_0)$ is always equal to 1 since we allow only simple designs.

We remark the following fact for the case $m > t$. Assume we know the intersection numbers $\alpha_{t+1}(M), \ldots, \alpha_m(M)$ (the *late* intersection numbers). Then, since the coefficient matrix has 1-s on the main diagonal one can easily compute the remaining numbers $\alpha_0(M), \ldots, \alpha_t(M)$ (the *early* intersection numbers). The terms *early* and *late* intersection numbers should not be mixed up with intersection numbers of *higher order* which will be introduced in the sequel. KÖHLER gives explicit equations for the early intersection numbers. In [7], he proves that

$$\begin{aligned} \alpha_i(M) \; &= \textstyle\sum_{h=i}^{t} (-1)^{h+i} \binom{h}{i} \binom{m}{h} \lambda_h \\ &+ \; (-1)^{t+i+1} \textstyle\sum_{h=0}^{m-t-1} \binom{t+h-i}{h} \binom{t+h+1}{i} \alpha_{t+h+1}(M), \end{aligned} \tag{8}$$

for $0 \leq i \leq t$.

For any $B_0 \in \mathcal{B}$, the vector $(\alpha_0(B_0), \ldots, \alpha_k(B_0))$ is called the *block intersection type* of $B_0$ (in the design $\mathcal{D}$). The equations of Köhler show that only the *essential block intersection* numbers are needed, that is $(\alpha_{t+1}(B_0), \ldots, \alpha_{k-1}(B_0))$. We will call this vector the *essential block intersection type*.

Clearly, block intersection types are constant on orbits of the automorphism group. So, when computing designs as orbits of some automorphism (sub-) group, we need only specify block intersection types for each of our orbit representatives. We will do so later when we specify the 8-$(31, 10, \lambda)$ designs as sets of orbits.

Let now $\mathcal{K} = \{K_1, \ldots, K_l\}$ be the representing sets for the $A$-orbits of blocks *in the design* (not to be mixed up with all $A$ orbits as in section 2). Let $\tilde{K_h}$ be the corresponding orbit under the action of $A$ $(h = 1, \ldots, l)$. Clearly, $\sum_{h=1}^{l} |\tilde{K_h}| = b$. We define the *global intersection number* as

$$\alpha_i(\mathcal{D}) \; = \; |\{\{B_u, B_v\} \in \binom{\mathcal{B}}{2} \mid |B_u \cap B_v| = i\}|, \tag{9}$$

By adding up the intersection types of all blocks of the design one gets the following formula – we count all intersections twice, therefore the factor $1/2$:

$$\alpha_i(\mathcal{D}) \;=\; \frac{1}{2}\sum_{B\in\mathcal{B}} \alpha_i(B) \;=\; \frac{1}{2}\sum_{h=1}^{l} |\tilde{K_h}|\cdot\alpha_i(K_h). \tag{10}$$

The vector $(\alpha_0(\mathcal{D}),\dots,\alpha_k(\mathcal{D}))$ is the *global intersection type of pairs of blocks* of the design. Clearly,

$$\sum_{i=0}^{k}\alpha_i(\mathcal{D}) \;=\; \binom{b}{2}, \tag{11}$$

but we will find more equations for global block intersection types in the following. In order to achieve this, let us introduce *intersection numbers of higher order* (already introduced by MENDELSOHN [10]).

For an arbitrary fixed $m$-subset $M$ of $V$ and $b \ge s \ge 1$ define for $0 \le i \le m$:

$$\alpha_i^{(s)}(M) \;:=\; |\{\{B_{j_1},\dots,B_{j_s}\}\in\binom{\mathcal{B}}{s} \mid |M\cap B_{j_1}\cap\ldots\cap B_{j_s}| = i\}|, \tag{12}$$

the $i$-th *intersection number of order* $s$ of $M$ with $\mathcal{D}$. In the case when $s = 1$, this reduces to ordinary intersection numbers. If $s$ is at least two and if $m$ is at least $k$, $\alpha_k^{(s)}(M) = 0$ for each $M \subseteq V$ as we have excluded designs with repeated blocks.

It can be shown (see TRAN VAN TRUNG, QIU-RONG WU, DALE M. MESNER [16]) that the following generalization of the equations of Mendelsohn holds (for an arbitrary $m$-subset $M$ of $V$, $b \ge s \ge 1$ and $0 \le j \le t$):

$$\sum_{i=j}^{m}\binom{i}{j}\alpha_i^{(s)}(M) = \binom{m}{j}\binom{\lambda_{j,0}}{s}. \tag{13}$$

The following generalization of Köhler's equations was also proved in [16]. Again, let $M$ be an (arbitrary) $m$-subset of $V$ and $0 \le i \le t$. Then, for each $b \ge s \ge 1$,

$$\begin{aligned}\alpha_i^{(s)}(M) \;=&\; \textstyle\sum_{h=i}^{t}(-1)^{h+i}\binom{h}{i}\binom{m}{h}\binom{\lambda_h}{s}\\ &+\;(-1)^{t+i+1}\textstyle\sum_{h=0}^{m-t-1}\binom{t+h-i}{h}\binom{t+h+1}{i}\alpha_{t+h+1}^{(s)}(M).\end{aligned} \tag{14}$$

For $s = 1$, these formulae reduce to (8), i.e. the equations for ordinary intersection numbers. Again, we see that only the *essential block intersection numbers* (of higher order) $(\alpha_{t+1}^{(s)}(B),\dots,\alpha_{k-1}^{(s)}(B))$ need to be specified (for $B$ a block of the design).

*Global intersection numbers of order* $s$ of the design $\mathcal{D}$ can be defined in the following way:

$$\alpha_i^{(s)}(\mathcal{D}) := |\{\{B_{j_1},\dots,B_{j_s}\}\in\binom{\mathcal{B}}{s} \mid |B_{j_1}\cap\ldots\cap B_{j_s}| = i\}|. \tag{15}$$

Clearly, in the case $s = 2$, we get back the values $\alpha_i(\mathcal{D})$ which we already know. Again, global intersection numbers can be computed by cumulating intersections over all block orbits:

$$\alpha_i^{(s)}(\mathcal{D}) \;=\; \frac{1}{s}\sum_{B\in\mathcal{B}} \alpha_i^{(s-1)}(B) \;=\; \frac{1}{s}\sum_{h=1}^{l} |\tilde{K_h}| \cdot \alpha_i^{(s-1)}(K_h). \tag{16}$$

These numbers can be checked in the following way: Choose $M = V$ and apply (13). This gives for $j = 0,\ldots,t$ and all $b \geq s \geq 1$:

$$\sum_{i=j}^{k} \binom{i}{j} \alpha_i^{(s)}(\mathcal{D}) = \binom{v}{j}\binom{\lambda_{j,0}}{s}. \tag{17}$$

To see this, one verifies that $\alpha_i^{(s)}(\mathcal{D}) = \alpha_i^{(s)}(V)$, by definition. We stopped the summation on the left after the $k$-th coefficient since clearly, $\alpha_i^{(s)}(\mathcal{D}) = 0$ for $i > k$ (moreover: $\alpha_k^{(s)}(\mathcal{D}) = 0$ for $s > 1$). In the case when $j = 0$ (and $s = 2$), we get back equation (11) – recall that $\alpha_i^{(2)}(\mathcal{D}) = \alpha_i(\mathcal{D})$.

Applying (14) with $M = V$ we are able to compute $\alpha_i^{(s)}(\mathcal{D})$ $(0 \leq i \leq t)$ from $(\alpha_{t+1}^{(s)}(\mathcal{D}),\ldots,\alpha_k^{(s)}(\mathcal{D}))$. The latter vector is called the *essential global block intersection type of order* $s$ of the design. For $s > 1$, $\alpha_k^{(s)}(\mathcal{D})$ vanishes.

## 5. 8-(31, 10, 100) Designs

### *5.1. Parameters and Intersection Equations*

The intersection triangle of $\lambda_{i,j}$ for $i + j \leq t$ is:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 17,530500 | 11,875500 | 7,917000 | 5,187000 | 3,334500 | 2,099500 | 1,292000 | 775200 | 452200 |
| 5,655000 | 3,958500 | 2,730000 | 1,852500 | 1,235000 | 807500 | 516800 | 323000 | |
| 1,696500 | 1,228500 | 877500 | 617500 | 427500 | 290700 | 193800 | | |
| 468000 | 351000 | 260000 | 190000 | 136800 | 96900 | | | |
| 117000 | 91000 | 70000 | 53200 | 39900 | | | | |
| 26000 | 21000 | 16800 | 13300 | | | | | |
| 5000 | 4200 | 3500 | | | | | | |
| 800 | 700 | | | | | | | |
| 100 | | | | | | | | |

(18)

The following values are helpful for the verification of some of the intersection numbers:

$$b^2 = 307,318430,250000$$
$$\binom{b}{2} = 153,659206,359750$$

$$\begin{aligned} b^3 &= 5387,445741,497625,000000 \\ \binom{b}{3} &= 897,907469,923728,218500. \end{aligned}$$

The system (7) of Mendelsohn for arbitrary $M \subseteq V$ of size $m = k = 10$ is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ & & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 \\ & & & 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120 \\ & & & & 1 & 5 & 15 & 35 & 70 & 126 & 210 \\ & & & & & 1 & 6 & 21 & 56 & 126 & 252 \\ & & & & & & 1 & 7 & 28 & 84 & 210 \\ & & & & & & & 1 & 8 & 36 & 120 \\ & & & & & & & & 1 & 9 & 45 \end{pmatrix} \begin{pmatrix} \alpha_0(M) \\ \alpha_1(M) \\ \alpha_2(M) \\ \alpha_3(M) \\ \alpha_4(M) \\ \alpha_5(M) \\ \alpha_6(M) \\ \alpha_7(M) \\ \alpha_8(M) \\ \alpha_9(M) \\ \alpha_{10}(M) \end{pmatrix} = \begin{pmatrix} 17,530500 \\ 56,550000 \\ 76,342500 \\ 56,160000 \\ 24,570000 \\ 6,552000 \\ 1,050000 \\ 96000 \\ 4500 \end{pmatrix} \quad (19)$$

These equations are important in particular if $M$ is a block $B_0$ of the design. In this case, the essential block intersection type consists of just one number, namely $\alpha_9(B_0)$. The equations (8) of Köhler for $m = k$ are:

$$\begin{aligned} \alpha_0(M) &= 139500 & -1\,\alpha_9(M) & & -9\,\alpha_{10}(M) \\ \alpha_1(M) &= 1,161000 & +9\,\alpha_9(M) & & +80\,\alpha_{10}(M) \\ \alpha_2(M) &= 3,622500 & -36\,\alpha_9(M) & & -315\,\alpha_{10}(M) \\ \alpha_3(M) &= 5,508000 & +84\,\alpha_9(M) & & +720\,\alpha_{10}(M) \\ \alpha_4(M) &= 4,515000 & -126\,\alpha_9(M) & & -1050\,\alpha_{10}(M) \\ \alpha_5(M) &= 2,016000 & +126\,\alpha_9(M) & & +1008\,\alpha_{10}(M) \\ \alpha_6(M) &= 504000 & -84\,\alpha_9(M) & & -630\,\alpha_{10}(M) \\ \alpha_7(M) &= 60000 & +36\,\alpha_9(M) & & +240\,\alpha_{10}(M) \\ \alpha_8(M) &= 4500 & -9\,\alpha_9(M) & & -45\,\alpha_{10}(M) \end{aligned} \quad (20)$$

If we consider generalized Mendelsohn systems (13), only the right hand side differs from the case $s = 1$. For the 8-(31,10,100) designs, we get the following vectors for $s = 2$ and $s = 3$:

$$\begin{pmatrix} 153,659206,359750 \\ 15,989509,672500 \\ 1,439055,276750 \\ 109511,766000 \\ 6844,441500 \\ 337,987000 \\ 12,497500 \\ 319600 \\ 4950 \end{pmatrix}, \begin{pmatrix} 897,907469,923728,218500 \\ 30,140215,072989,385000 \\ 813784,799631,940500 \\ 17083,762488,156000 \\ 266,928655,539000 \\ 2,928995,342000 \\ 20820,835000 \\ 85,013600 \\ 161700 \end{pmatrix}. \quad (21)$$

Next, we evaluate the generalized Köhler equations (14). Choosing $V = M$ and using the equalities $\alpha_i^{(s)}(\mathcal{D}) = \alpha_i^{(s)}(V)$ and $\alpha_k^{(s)}(M) = 0$ for $s \geq 2$ we get:

$$\begin{array}{lcrr}
\alpha_0^{(2)}(\mathcal{D}) & = & 1,222673,487750 & -1\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_1^{(2)}(\mathcal{D}) & = & 10,177156,470000 & +9\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_2^{(2)}(\mathcal{D}) & = & 31,749357,071250 & -36\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_3^{(2)}(\mathcal{D}) & = & 48,285307,980000 & +84\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_4^{(2)}(\mathcal{D}) & = & 39,565900,237500 & -126\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_5^{(2)}(\mathcal{D}) & = & 17,679579,372000 & +126\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_6^{(2)}(\mathcal{D}) & = & 4,412163,892500 & -84\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_7^{(2)}(\mathcal{D}) & = & 528018,660000 & +36\,\alpha_9^{(2)}(\mathcal{D}) \\
\alpha_8^{(2)}(\mathcal{D}) & = & 39049,188750 & -9\,\alpha_9^{(2)}(\mathcal{D})
\end{array} \tag{22}$$

For $s = 3$ we find:

$$\begin{array}{lcrr}
\alpha_0^{(3)}(\mathcal{D}) & = & 273,095571,435812,716500 & -1\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_1^{(3)}(\mathcal{D}) & = & 376,703617,540702,270000 & +9\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_2^{(3)}(\mathcal{D}) & = & 193,677461,232560,077500 & -36\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_3^{(3)}(\mathcal{D}) & = & 47,873710,172695,680000 & +84\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_4^{(3)}(\mathcal{D}) & = & 6,132782,342156,925000 & -126\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_5^{(3)}(\mathcal{D}) & = & 410311,724665,752000 & +126\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_6^{(3)}(\mathcal{D}) & = & 13800,854745,405000 & -84\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_7^{(3)}(\mathcal{D}) & = & 213,344782,560000 & +36\,\alpha_9^{(3)}(\mathcal{D}) \\
\alpha_8^{(3)}(\mathcal{D}) & = & 1,275606,832500 & -9\,\alpha_9^{(3)}(\mathcal{D})
\end{array} \tag{23}$$

### *5.2. The Designs*

We display 3 out of the 1658 designs for $\lambda = 100$. The designs are collections of full orbits from the list of 10-orbits of the group. Here, we list only the orbit numbers using the labelling of orbits of section 2.

$\mathfrak{D}_1$ : 6, 7, 9, 10, 14, 15, 18, 23, 25, 29, 31, 34, 37, 38, 39, 40, 47, 49, 50, 56, 59, 60, 64, 66, 67, 68, 70, 72, 76, 77, 78, 79, 84, 87, 88, 91, 92, 94, 96, 100, 102, 105, 106, 108, 113, 114, 117, 118, 120, 121, 124, 126, 136, 139, 141, 143, 145, 147, 148, 149, 151, 153, 156, 157, 160, 164, 165, 171, 173.

Block intersection types:

$\alpha_9(B_h) = 72$ for $h \in \{149\}$, $\alpha_9(B_h) = 73$ for $h \in \{141, 171\}$, $\alpha_9(B_h) = 75$ for $h \in \{25, 147\}$, $\alpha_9(B_h) = 76$ for $h \in \{56, 87, 91, 102\}$, $\alpha_9(B_h) = 77$ for $h \in \{18, 49, 60, 78, 118\}$, $\alpha_9(B_h) = 78$ for $h \in \{34, 106, 117, 120, 136, 139\}$, $\alpha_9(B_h) = 79$ for $h \in \{9, 10, 64, 68, 76, 77, 94, 96, 124, 126, 164, 165\}$, $\alpha_9(B_h) = 80$ for $h \in \{6, 29, 38, 84, 113, 121\}$, $\alpha_9(B_h) = 81$ for $h \in \{37, 50, 66, 67, 70, 92, 108, 145, 160, 173\}$, $\alpha_9(B_h) = 82$ for $h \in \{7, 39, 59, 105, 143, 151\}$, $\alpha_9(B_h) = 83$ for $h \in \{23, 72, 79, 88, 114, 148, 153\}$, $\alpha_9(B_h) = 84$ for $h \in \{31, 100, 156\}$, $\alpha_9(B_h) = 85$ for $h \in \{14, 47\}$, $\alpha_9(B_h) = 87$ for $h \in \{15, 40, 157\}$.

The following table shows the global intersection numbers of all 2-sets of blocks (all 3-sets of blocks). The column sums are $\binom{b}{2}$ and $\binom{b}{3}$ respectively. The values of these tables contain a lot of redundancy. According to (14), only $\alpha_9^{(2)}(\mathcal{D})$ and $\alpha_9^{(3)}(\mathcal{D})$ really matters. The other values follow. In fact, all the numbers in these tables have been computed from the orbit data. So, verification of (14) via (22) and (23) really is a good test for the correctness of our algorithms to compute intersection numbers.

| $i$ | $\alpha_i^{(2)}(\mathcal{D})$ | $\alpha_i^{(3)}(\mathcal{D})$ |
|---|---|---|
| 0 | 1,221974,151000 | 273,095571,434113,653000 |
| 1 | 10,183450,500750 | 376,703617,555993,841500 |
| 2 | 31,724180,948250 | 193,677461,171393,791500 |
| 3 | 48,344052,267000 | 47,873710,315417,014000 |
| 4 | 39,477783,807000 | 6,132782,128074,924000 |
| 5 | 17,767695,802500 | 410311,938747,753000 |
| 6 | 4,353419,605500 | 13800,712024,071000 |
| 7 | 553194,783000 | 213,405948,846000 |
| 8 | 32755,158000 | 1,260315,261000 |
| 9 | 699,336750 | 1699,063500 |
| 10 | 0 | 0 |
| $\sum$ | 153,659206,359750 | 897,907469,923728,218500 |

The number $\alpha_9^{(2)}(\mathcal{D})$ can be computed according to (10) as the following sum. The pairs of numbers of the form "$a \times b$" give the multiplicity ($a$) together with the intersection number $\alpha_9$ ($b$). The greatest common divisor of all the multiplicities is taken out of the sum.

$699,336750 = \frac{1}{2} \cdot 15500 \cdot (6 \times 72 + 18 \times 73 + 30 \times 75 + 78 \times 76 + 72 \times 77 + 128 \times 78 + 210 \times 79 + 132 \times 80 + 168 \times 81 + 96 \times 82 + 114 \times 83 + 48 \times 84 + 18 \times 85 + 13 \times 87)$

$\mathfrak{D}_2$ : 5, 7, 10, 11, 14, 15, 17, 24, 25, 27, 28, 29, 30, 31, 39, 40, 45, 47, 49, 52, 56, 58, 59, 60, 62, 65, 69, 71, 76, 77, 78, 80, 83, 86, 87, 88, 95, 98, 99, 100, 103, 104, 105, 106, 110, 112, 113, 120, 126, 130, 131, 132, 134, 141, 142, 144, 145, 146, 148, 149, 151, 153, 156, 157, 160, 164, 170, 171, 173.

Block intersection types:

$\alpha_9(B_h) = 74$ for $h \in \{56, 77, 141\}$, $\alpha_9(B_h) = 75$ for $h \in \{131\}$, $\alpha_9(B_h) = 76$ for $h \in \{58, 69\}$, $\alpha_9(B_h) = 77$ for $h \in \{28, 76, 78, 99, 104, 126, 148\}$, $\alpha_9(B_h) = 78$ for $h \in \{24, 47, 52, 71, 86, 106, 110, 151\}$, $\alpha_9(B_h) = 79$ for $h \in \{7, 49, 134, 153, 164\}$, $\alpha_9(B_h) = 80$ for $h \in \{17, 29, 62, 83, 95, 98, 105, 113, 142, 170\}$, $\alpha_9(B_h) = 81$ for $h \in \{15, 87, 88, 132, 145, 157, 171\}$, $\alpha_9(B_h) = 82$ for $h \in \{5, 27, 31, 80, 103, 120, 149, 156\}$, $\alpha_9(B_h) = 83$ for $h \in \{10, 30, 39, 40, 65, 112, 144, 146\}$, $\alpha_9(B_h) = 84$ for $h \in \{45, 59, 100, 130, 173\}$, $\alpha_9(B_h) = 85$ for $h \in \{11\}$, $\alpha_9(B_h) = 87$ for $h \in \{160\}$, $\alpha_9(B_h) = 88$ for $h \in \{60\}$, $\alpha_9(B_h) = 89$ for $h \in \{14, 25\}$.

Global intersections:

| $i$ | $\alpha_i^{(2)}(\mathcal{D})$ | $\alpha_i^{(3)}(\mathcal{D})$ |
|---|---|---|
| 0 | 1,221971,547000 | 273,095571,434098,029000 |
| 1 | 10,183473,936750 | 376,703617,556134,457500 |
| 2 | 31,724087,204250 | 193,677461,170831,327500 |
| 3 | 48,344271,003000 | 47,873710,316729,430000 |
| 4 | 39,477455,703000 | 6,132782,126106,300000 |
| 5 | 17,768023,906500 | 410311,940716,377000 |
| 6 | 4,353200,869500 | 13800,710711,655000 |
| 7 | 553288,527000 | 213,406511,310000 |
| 8 | 32731,722000 | 1,260174,645000 |
| 9 | 701,940750 | 1714,687500 |
| 10 | 0 | 0 |
| $\sum$ | 153,659206,359750 | 897,907469,923728,218500 |

$701,940750 = \frac{1}{2}\cdot 15500\cdot(42\times 74+24\times 75+36\times 76+132\times 77+116\times 78+84\times 79+216\times 80+112\times 81+138\times 82+123\times 83+68\times 84+12\times 85+4\times 87+12\times 88+12\times 89)$

$\mathfrak{D}_3$ : 5, 8, 9, 10, 14, 15, 18, 19, 24, 25, 28, 29, 31, 34, 37, 39, 40, 45, 47, 50, 53, 56, 59, 61, 63, 65, 68, 71, 72, 73, 80, 83, 91, 94, 99, 100, 103, 104, 106, 109, 110, 111, 112, 116, 117, 119, 120, 125, 128, 129, 130, 132, 133, 134, 140, 141, 143, 144, 146, 148, 149, 153, 157, 160, 163, 164, 170, 171, 173.
Block intersection types:
$\alpha_9(B_h) = 73$ for $h \in \{25, 153\}$, $\alpha_9(B_h) = 74$ for $h \in \{24\}$, $\alpha_9(B_h) = 75$ for $h \in \{15, 37, 117, 173\}$, $\alpha_9(B_h) = 76$ for $h \in \{104, 120, 133\}$, $\alpha_9(B_h) = 77$ for $h \in \{10, 18, 19, 47, 71, 109, 132, 171\}$, $\alpha_9(B_h) = 78$ for $h \in \{61, 80, 91, 129, 149, 160\}$, $\alpha_9(B_h) = 79$ for $h \in \{8, 29, 40, 72, 103, 112, 125, 128, 157\}$, $\alpha_9(B_h) = 80$ for $h \in \{5, 34, 53, 63, 65, 83, 94, 100, 110, 119, 134, 140\}$, $\alpha_9(B_h) = 81$ for $h \in \{28, 50, 68, 99, 106, 111, 144, 148, 163\}$, $\alpha_9(B_h) = 82$ for $h \in \{116, 130, 141\}$, $\alpha_9(B_h) = 83$ for $h \in \{9, 170\}$, $\alpha_9(B_h) = 84$ for $h \in \{31, 39, 146\}$, $\alpha_9(B_h) = 85$ for $h \in \{14, 59, 143\}$, $\alpha_9(B_h) = 86$ for $h \in \{45, 56, 73\}$, $\alpha_9(B_h) = 87$ for $h \in \{164\}$.
Global intersections:

| $i$ | $\alpha_i^{(2)}(\mathcal{D})$ | $\alpha_i^{(3)}(\mathcal{D})$ |
|---|---|---|
| 0 | 1,221975,360000 | 273,095571,434121,651000 |
| 1 | 10,183439,619750 | 376,703617,555921,859500 |
| 2 | 31,724224,472250 | 193,677461,171681,719500 |
| 3 | 48,343950,711000 | 47,873710,314745,182000 |
| 4 | 39,477936,141000 | 6,132782,129082,672000 |
| 5 | 17,767543,468500 | 410311,937740,005000 |
| 6 | 4,353521,161500 | 13800,712695,903000 |
| 7 | 553151,259000 | 213,405660,918000 |
| 8 | 32766,039000 | 1,260387,243000 |
| 9 | 698,127750 | 1691,065500 |
| 10 | 0 | 0 |
| $\sum$ | 153,659206,359750 | 897,907469,923728,218500 |

$698,127750 = \frac{1}{2}\cdot 15500\cdot(12\times 73+12\times 74+60\times 75+72\times 76+114\times 77+106\times 78+129\times 79+252\times 80+164\times 81+48\times 82+24\times 83+48\times 84+42\times 85+42\times 86+6\times 87)$

## 6. 8-(31, 10, 93) Designs

### *6.1. Parameters and Intersection Equations*

Again, we list 3 of the designs, now with $\lambda = 93$. We have $\lambda_{i,j} =$

$$
\begin{array}{rrrrrrrrr}
16,303365 & 11,044215 & 7,362810 & 4,823910 & 3,101085 & 1,952535 & 1,201560 & 720936 & 420546\\
5,259150 & 3,681405 & 2,538900 & 1,722825 & 1,148550 & 750975 & 480624 & 300390 & \\
1,577745 & 1,142505 & 816075 & 574275 & 397575 & 270351 & 180234 & & \\
435240 & 326430 & 241800 & 176700 & 127224 & 90117 & & & \\
108810 & 84630 & 65100 & 49476 & 37107 & & & & \\
24180 & 19530 & 15624 & 12369 & & & & & \\
4650 & 3906 & 3255 & & & & & & \\
744 & 651 & & & & & & & \\
93 & & & & & & & &
\end{array}
\tag{24}
$$

Some useful values are:

$$
\begin{aligned}
b^2 &= 265,799710,323225\\
\binom{b}{2} &= 132,899847,009930\\
b^3 &= 4333,429694,293805,152125\\
\binom{b}{3} &= 722,238149,482451,131530.
\end{aligned}
$$

The system of Mendelsohn is:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
 & & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45\\
 & & & 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120\\
 & & & & 1 & 5 & 15 & 35 & 70 & 126 & 210\\
 & & & & & 1 & 6 & 21 & 56 & 126 & 252\\
 & & & & & & 1 & 7 & 28 & 84 & 210\\
 & & & & & & & 1 & 8 & 36 & 120\\
 & & & & & & & & 1 & 9 & 45
\end{pmatrix}
\begin{pmatrix}
\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\\ \alpha_7\\ \alpha_8\\ \alpha_9\\ \alpha_{10}
\end{pmatrix}
=
\begin{pmatrix}
16303365\\ 52591500\\ 70998525\\ 52228800\\ 22850100\\ 6093360\\ 976500\\ 89280\\ 4185
\end{pmatrix}
\tag{25}
$$

The equations of Köhler are (for $M \subseteq V$, $|M| = 10$):

$$
\begin{array}{rcrrr}
\alpha_0(M) & = & 129735 & -1\,\alpha_9(M) & -9\,\alpha_{10}(M)\\
\alpha_1(M) & = & 1,079730 & +9\,\alpha_9(M) & +80\,\alpha_{10}(M)\\
\alpha_2(M) & = & 3,368925 & -36\,\alpha_9(M) & -315\,\alpha_{10}(M)\\
\alpha_3(M) & = & 5,122440 & +84\,\alpha_9(M) & +720\,\alpha_{10}(M)\\
\alpha_4(M) & = & 4,198950 & -126\,\alpha_9(M) & -1050\,\alpha_{10}(M)\\
\alpha_5(M) & = & 1,874880 & +126\,\alpha_9(M) & +1008\,\alpha_{10}(M)\\
\alpha_6(M) & = & 468720 & -84\,\alpha_9(M) & -630\,\alpha_{10}(M)\\
\alpha_7(M) & = & 55800 & +36\,\alpha_9(M) & +240\,\alpha_{10}(M)\\
\alpha_8(M) & = & 4185 & -9\,\alpha_9(M) & -45\,\alpha_{10}(M)
\end{array}
\tag{26}
$$

The generalized Mendelsohn systems (13) have the following right hand side (for $s = 2$ and $s = 3$)

$$\begin{pmatrix} 132,899847,009930 \\ 13,829326,731675 \\ 1,244638,853640 \\ 94716,711180 \\ 5919,753645 \\ 292,324110 \\ 10,808925 \\ 276396 \\ 4278 \end{pmatrix}, \begin{pmatrix} 722,238149,482451,131530 \\ 24,243492,007411,704300 \\ 654573,412952,844840 \\ 13741,437313,520280 \\ 214,705518,201720 \\ 2,355937,443860 \\ 16746,627800 \\ 68,361944 \\ 129766 \end{pmatrix}. \tag{27}$$

The generalized KÖHLER equations applied to $M = V$ (with $m = v$) and $s = 2$ and $s = 3$ are (the $\alpha_j^{(s)}(\mathcal{D})$-terms with $j = k, \ldots, v$ left out):

$$\begin{array}{lcrr} \alpha_0^{(2)}(\mathcal{D}) & = & 1,057485,163995 & -1\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_1^{(2)}(\mathcal{D}) & = & 8,802268,280325 & +9\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_2^{(2)}(\mathcal{D}) & = & 27,459839,186325 & -36\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_3^{(2)}(\mathcal{D}) & = & 41,762373,716700 & +84\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_4^{(2)}(\mathcal{D}) & = & 34,219947,966750 & -126\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_5^{(2)}(\mathcal{D}) & = & 15,291643,381560 & +126\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_6^{(2)}(\mathcal{D}) & = & 3,815721,061425 & -84\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_7^{(2)}(\mathcal{D}) & = & 456820,287300 & +36\,\alpha_9^{(2)}(\mathcal{D}) \\ \alpha_8^{(2)}(\mathcal{D}) & = & 33747,965550 & -9\,\alpha_9^{(2)}(\mathcal{D}) \end{array} \tag{28}$$

$$\begin{array}{lcrr} \alpha_0^{(3)}(\mathcal{D}) & = & 219,666334,479373,519920 & -1\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_1^{(3)}(\mathcal{D}) & = & 303,004191,078014,790000 & +9\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_2^{(3)}(\mathcal{D}) & = & 155,785819,762501,110000 & -36\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_3^{(3)}(\mathcal{D}) & = & 38,507550,969895,043400 & +84\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_4^{(3)}(\mathcal{D}) & = & 4,932944,011165,716000 & -126\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_5^{(3)}(\mathcal{D}) & = & 330036,037337,047860 & +126\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_6^{(3)}(\mathcal{D}) & = & 11100,547123,029000 & -84\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_7^{(3)}(\mathcal{D}) & = & 171,573352,587000 & +36\,\alpha_9^{(3)}(\mathcal{D}) \\ \alpha_8^{(3)}(\mathcal{D}) & = & 1,023688,288350 & -9\,\alpha_9^{(3)}(\mathcal{D}) \end{array} \tag{29}$$

### *6.2. The Designs*

$\mathfrak{D}_1$ : 1, 2, 5, 7, 9, 12, 13, 14, 16, 19, 24, 25, 29, 30, 33, 36, 39, 42, 43, 46, 48, 52, 53, 55, 57, 60, 64, 65, 72, 75, 76, 81, 83, 84, 85, 91, 92, 94, 96, 98, 103, 105, 107, 109, 113, 114,

116, 120, 124, 125, 126, 128, 131, 132, 136, 138, 139, 141, 147, 148, 149, 150, 152, 159, 162, 167, 168, 172.
Block intersection types:
$\alpha_9(B_h) = 10$ for $h \in \{1\}$, $\alpha_9(B_h) = 60$ for $h \in \{172\}$, $\alpha_9(B_h) = 66$ for $h \in \{148, 167\}$, $\alpha_9(B_h) = 68$ for $h \in \{9, 36, 149\}$, $\alpha_9(B_h) = 69$ for $h \in \{159\}$, $\alpha_9(B_h) = 70$ for $h \in \{128\}$, $\alpha_9(B_h) = 71$ for $h \in \{25, 30, 65, 76, 96, 126\}$, $\alpha_9(B_h) = 72$ for $h \in \{5, 42, 46, 52, 60, 75, 81, 85, 103, 116, 120, 124, 132\}$, $\alpha_9(B_h) = 73$ for $h \in \{98, 105, 141\}$, $\alpha_9(B_h) = 74$ for $h \in \{53, 57, 83, 91, 92, 138, 139, 147, 150, 168\}$, $\alpha_9(B_h) = 75$ for $h \in \{16, 24, 29, 33, 64, 94, 136, 152, 162\}$, $\alpha_9(B_h) = 76$ for $h \in \{72, 109, 114, 125\}$, $\alpha_9(B_h) = 77$ for $h \in \{7, 39, 107\}$, $\alpha_9(B_h) = 79$ for $h \in \{14, 19, 43, 48, 84, 113, 131\}$, $\alpha_9(B_h) = 80$ for $h \in \{55\}$, $\alpha_9(B_h) = 82$ for $h \in \{12\}$, $\alpha_9(B_h) = 84$ for $h \in \{13\}$, $\alpha_9(B_h) = 90$ for $h \in \{2\}$.
Global intersections:

| $i$ | $\alpha_i^{(2)}(\mathcal{D})$ | $\alpha_i^{(3)}(\mathcal{D})$ |
|---|---|---|
| 0 | 1,056882,079920 | 219,666334,478026,507920 |
| 1 | 8,807696,037000 | 303,004191,090137,898000 |
| 2 | 27,438128,159625 | 155,785819,714008,678000 |
| 3 | 41,813032,779000 | 38,507551,083044,051400 |
| 4 | 34,143959,373300 | 4,932943,841442,204000 |
| 5 | 15,367631,975010 | 330036,207060,559860 |
| 6 | 3,765061,999125 | 11100,433974,021000 |
| 7 | 478531,314000 | 171,621845,019000 |
| 8 | 28320,208875 | 1,011565,180350 |
| 9 | 603,084075 | 1347,012000 |
| 10 | 0 | 0 |
| $\sum$ | 132,899847,009930 | 722,238149,482451,131530 |

$603,084075 = \frac{1}{2} \cdot 155 \cdot (3 \times 10 + 100 \times 60 + 1400 \times 66 + 3000 \times 68 + 400 \times 69 + 1200 \times 70 + 11400 \times 71 + 22800 \times 72 + 4800 \times 73 + 17400 \times 74 + 15800 \times 75 + 7200 \times 76 + 6000 \times 77 + 12600 \times 79 + 150 \times 80 + 600 \times 82 + 300 \times 84 + 30 \times 90)$

$\mathfrak{D}_2$ : 1, 2, 5, 7, 9, 12, 13, 14, 16, 19, 24, 25, 29, 30, 35, 36, 39, 42, 43, 46, 49, 52, 53, 55, 57, 60, 63, 69, 70, 72, 75, 78, 80, 84, 85, 90, 94, 95, 98, 100, 101, 103, 104, 105, 110, 116, 117, 121, 122, 125, 128, 130, 134, 135, 137, 138, 139, 143, 147, 148, 149, 152, 156, 159, 163, 167, 169, 170, 172.
Block intersection types:
$\alpha_9(B_h) = 10$ for $h \in \{1\}$, $\alpha_9(B_h) = 48$ for $h \in \{55\}$, $\alpha_9(B_h) = 68$ for $h \in \{36, 46\}$, $\alpha_9(B_h) = 69$ for $h \in \{143, 152\}$, $\alpha_9(B_h) = 70$ for $h \in \{84, 94\}$, $\alpha_9(B_h) = 71$ for $h \in \{14, 30\}$, $\alpha_9(B_h) = 72$ for $h \in \{49, 75, 95, 103, 104, 121, 159, 169\}$, $\alpha_9(B_h) = 73$ for $h \in \{63, 72, 90, 116, 130, 139, 170\}$, $\alpha_9(B_h) = 74$ for $h \in \{9, 12, 16, 29, 52, 53, 85, 117, 122, 125, 128, 147, 149, 156\}$, $\alpha_9(B_h) = 75$ for $h \in \{7, 43, 70, 78, 101, 105, 110\}$, $\alpha_9(B_h) = 76$ for $h \in \{5, 35, 60, 98\}$, $\alpha_9(B_h) = 77$ for $h \in \{24, 39, 100, 163\}$, $\alpha_9(B_h) = 78$ for $h \in \{80, 135, 167\}$, $\alpha_9(B_h) = 79$ for $h \in \{19, 25, 57, 69, 134, 137\}$, $\alpha_9(B_h) = 80$ for $h \in \{148\}$, $\alpha_9(B_h) = 84$ for $h \in \{42, 172\}$, $\alpha_9(B_h) = 86$ for $h \in \{138\}$, $\alpha_9(B_h) = 90$ for $h \in \{2\}$, $\alpha_9(B_h) = 92$ for $h \in \{13\}$.
Global intersections:

| $i$ | $\alpha_i^{(2)}(\mathcal{D})$ | $\alpha_i^{(3)}(\mathcal{D})$ |
|---|---|---|
| 0 | 1,056877,987920 | 219,666334,477992,531920 |
| 1 | 8,807732,865000 | 303,004191,090443,682000 |
| 2 | 27,437980,847625 | 155,785819,712785,542000 |
| 3 | 41,813376,507000 | 38,507551,085898,035400 |
| 4 | 34,143443,781300 | 4,932943,837161,228000 |
| 5 | 15,368147,567010 | 330036,211341,535860 |
| 6 | 3,764718,271125 | 11100,431120,037000 |
| 7 | 478678,626000 | 171,623068,155000 |
| 8 | 28283,380875 | 1,011259,396350 |
| 9 | 607,176075 | 1380,988000 |
| 10 | 0 | 0 |
| $\sum$ | 132,899847,009930 | 722,238149,482451,131530 |

$607,176075 = \frac{1}{2} \cdot 155 \cdot (3 \times 10 + 150 \times 48 + 1800 \times 68 + 1800 \times 69 + 4800 \times 70 + 3000 \times 71 + 14400 \times 72 + 13200 \times 73 + 22800 \times 74 + 12000 \times 75 + 7200 \times 76 + 4800 \times 77 + 5000 \times 78 + 11400 \times 79 + 1200 \times 80 + 700 \times 84 + 600 \times 86 + 30 \times 90 + 300 \times 92)$

$\mathfrak{D}_3$ : 1, 2, 5, 8, 11, 12, 13, 14, 17, 19, 24, 25, 28, 29, 33, 36, 39, 42, 43, 46, 47, 48, 52, 55, 58, 60, 62, 64, 66, 71, 75, 76, 77, 78, 81, 85, 88, 90, 94, 97, 98, 105, 109, 111, 113, 116, 118, 119, 120, 125, 126, 127, 131, 132, 133, 136, 138, 140, 145, 146, 149, 151, 152, 159, 167, 169, 172.
Block intersection types: $\alpha_9(B_h) = 10$ for $h \in \{1\}$, $\alpha_9(B_h) = 48$ for $h \in \{172\}$, $\alpha_9(B_h) = 60$ for $h \in \{159\}$, $\alpha_9(B_h) = 66$ for $h \in \{167\}$, $\alpha_9(B_h) = 67$ for $h \in \{152\}$, $\alpha_9(B_h) = 68$ for $h \in \{77\}$, $\alpha_9(B_h) = 69$ for $h \in \{25, 47, 98\}$, $\alpha_9(B_h) = 70$ for $h \in \{17, 109\}$, $\alpha_9(B_h) = 71$ for $h \in \{24, 60, 75, 126, 136\}$, $\alpha_9(B_h) = 72$ for $h \in \{19, 42, 55, 58, 81, 90, 140, 149, 151\}$, $\alpha_9(B_h) = 73$ for $h \in \{66, 78, 127, 131\}$, $\alpha_9(B_h) = 74$ for $h \in \{11, 12, 36, 43, 46, 85, 111, 119, 125, 146\}$, $\alpha_9(B_h) = 75$ for $h \in \{8, 14, 28, 33, 76, 88, 116, 120, 132, 145\}$, $\alpha_9(B_h) = 76$ for $h \in \{71, 94, 118, 138\}$, $\alpha_9(B_h) = 77$ for $h \in \{5, 52, 97, 133\}$, $\alpha_9(B_h) = 78$ for $h \in \{29, 39, 48, 62, 105, 113, 169\}$, $\alpha_9(B_h) = 81$ for $h \in \{64\}$, $\alpha_9(B_h) = 84$ for $h \in \{13\}$, $\alpha_9(B_h) = 90$ for $h \in \{2\}$.

Global intersections:

| $i$ | $\alpha_i^{(2)}(\mathcal{D})$ | $\alpha_i^{(3)}(\mathcal{D})$ |
|---|---|---|
| 0 | 1,056883,009920 | 219,666334,478018,912920 |
| 1 | 8,807687,667000 | 303,004191,090206,253000 |
| 2 | 27,438161,639625 | 155,785819,713735,258000 |
| 3 | 41,812954,659000 | 38,507551,083682,031400 |
| 4 | 34,144076,553300 | 4,932943,840485,234000 |
| 5 | 15,367514,795010 | 330036,208017,529860 |
| 6 | 3,765140,119125 | 11100,433336,041000 |
| 7 | 478497,834000 | 171,622118,439000 |
| 8 | 28328,578875 | 1,011496,825350 |
| 9 | 602,154075 | 1354,607000 |
| 10 | 0 | 0 |
| $\sum$ | 132,899847,009930 | 722,238149,482451,131530 |

$602,154075 = \frac{1}{2} \cdot 155 \cdot (3 \times 10 + 100 \times 48 + 400 \times 60 + 200 \times 66 + 600 \times 67 + 2400 \times 68 + 4200 \times 69 + 4800 \times 70 + 9600 \times 71 + 12150 \times 72 + 8400 \times 73 + 14400 \times 74 + 21000 \times 75 + 5400 \times 76 + 7200 \times 77 + 12800 \times 78 + 1200 \times 81 + 300 \times 84 + 30 \times 90)$

## 7. Isomorphism Problems

This section addresses problem 1 of section 1. We answer the question posed there by showing that all designs are non-isomorphic. This claim is proved in two different ways.

### *7.1. First Proof*

General group theoretic tools quite often suffice to solve the isomorphism problem for the designs constructed by the Kramer-Mesner method. This approach was already partly used in [12], [13] and we first briefly report the basic idea from that papers. Let $S_V$ be the full symmetric group on the underlying point set $V$. The following Lemma is useful when constructing objects with a prescribed automorphism group.

LEMMA 1 *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be designs with a group $A$ as their (full) group of automorphisms. Assume that $g \in S_V$ maps $\mathcal{D}_1$ onto $\mathcal{D}_2$. Then $g$ belongs to the normalizer of $A$ in $S_V$.*

**Proof:** $A = \mathrm{Stab}_{S_V}(\mathcal{D}_2) = \mathrm{Stab}_{S_V}(\mathcal{D}_1^g) = \mathrm{Stab}_{S_V}(\mathcal{D}_1)^g = A^g$. ∎

If the prescribed group of automorphisms $A$ is a maximal subgroup of $S_V$ different from the alternating group then all designs found are pairwise non-isomorphic.

If $A$ is not a maximal subgroup one can apply a Moebius inversion on the subgroup lattice to single out those designs having $A$ as their full automorphism group and then form the $N_{S_V}(A)$ orbits on the set of these designs. These orbits, all of length $|N_{S_V}(A)/A|$, are just the different isomorphism types.

A severe drawback of this approach is that it relies on the knowledge of the set of groups containing $A$ in $S_V$. Often the information on overgroups can be obtained in some way from the classification of the finite simple groups. We want to show here that in important cases we can avoid this laborious task by a localization technique. We regard $A$ as a guess for the automorphism group of the designs constructed. A good guess might at least find a correct Sylow subgroup of the automorphism group. Then the following holds.

LEMMA 2 *Let a finite group $G$ act on a set $\Omega$. Let $\omega_1, \omega_2 \in \Omega$ be fixed by a $p$-subgroup $P$ of $G$ and $g \in G$ such that $\omega_1^g = \omega_2$. Let $P$ be a Sylow subgroup of* $\mathrm{Stab}_G(\omega_2)$. *Then $\omega_1^n = \omega_2$ for some $n \in N_G(P)$.*

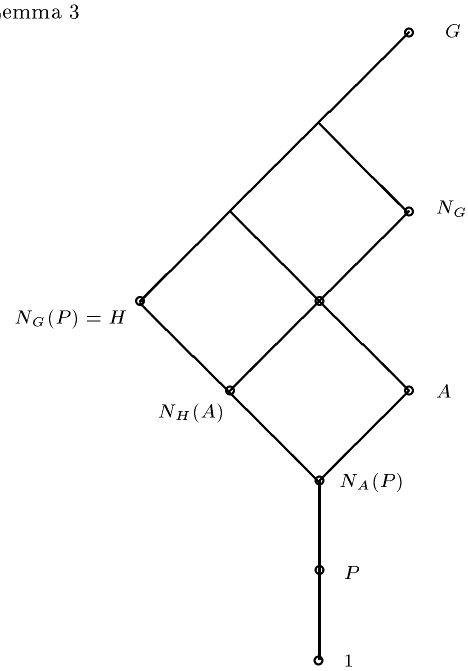
**Proof:** Since $P^g \leq \mathrm{Stab}_G(\omega_1)^g = \mathrm{Stab}_G(\omega_1^g) = \mathrm{Stab}_G(\omega_2)$ and $P \leq \mathrm{Stab}_G(\omega_2)$, there is some $x \in N_G(\omega_2)$ such that $P^g = P^x$ by the Sylow Theorem. Then $gx^{-1} \in N_G(P)$ and $g = nx$. Therefore $\omega_2 = \omega_1^g = \omega_1^{nx}$ and $\omega_1^n = \omega_2^{x^{-1}} = \omega_2$. ∎

If the prescribed subgroup $A$ of the automorphism group of the objects that are searched for contains a Sylow subgroup $P$ of all the automorphism groups of the objects then only elements from $N_G(P)$ have to be applied to the objects as possible isomorphisms. In our applications to $t$-designs it is often possible to show that no design exists if the proposed subgroup $P$ is extended to a larger $p$-group. Then the assumptions of the Lemma are of course fulfilled.

There is a problem if $A$ is not normalized by $N_G(P)$. Then, usually, the set of fixed points of $A$ is not closed under $N_G(P)$. So we cannot just form the orbits of $N_G(P)$ in order to solve the isomorphism problem.

LEMMA 3 *Let $G$ be a finite group acting on a set $\Omega$ and $A \leq G$. Let $A$ contain a Sylow subgroup $P$ of all designs admitting $A$ as a group of automorphisms. If $H = N_G(P)$ then $N_H(A)$ acts on the set of fixed points* $\mathrm{Fix}_\Omega(A)$ *of $A$ in $\Omega$. If $\omega_1^g = \omega_2$ for $\omega_1, \omega_2 \in \mathrm{Fix}_\Omega(A)$ and $g \in H$ with $N_H(A)g \neq N_H(A)$ then $A < \langle A, A^g \rangle \leq \mathrm{Stab}_G(\omega_2)$.*

*Figure 1.* Subgroups of Lemma 3



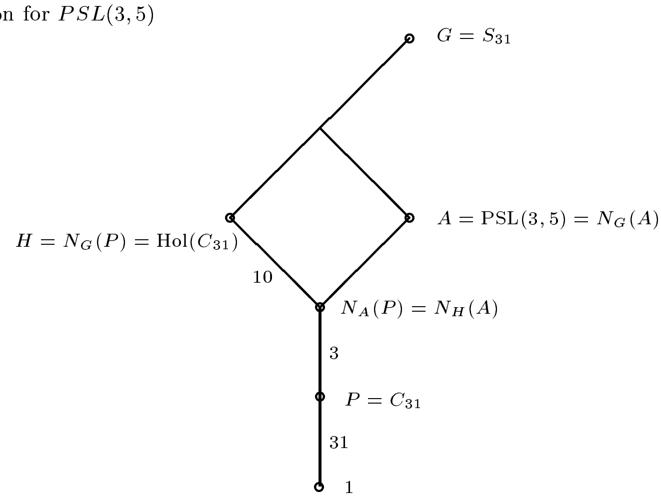

Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two designs admitting $A$ as an automorphism group. If $A$ is a large subgroup of the full symmetric group $S_n$ then it happens very often that $\langle A, A^g \rangle = S_n$ or $A_n$.

If this situation appears for each $g \in H \setminus N_H(A)$ then the orbits of $N_H(A)$ on $Fix_\Omega(A)$ are the different isomorphism types appearing in $Fix_\Omega(A)$. If in addition $N_H(A) \leq A$ then all designs admitting $A$ as an automorphism group are pairwise non-isomorphic. Algorithmically, only representatives from the cosets $N_H(A)g$ in $H$ have to be considered in forming $\langle A, A^g\rangle$.

A remarkable feature of this approach is that the individual designs are not touched upon. So, the isomorphism problem may be solved without knowing details like orbit representatives etc. of the designs.

To solve the isomorphism problem for the 8-designs of this paper we use the fact that $A = \mathrm{PSL}(3,5)$ contains a Sylow-31 subgroup $P$ of $S_{31}$ (cf. figure 2).

*Figure 2.* Special Situation for $PSL(3,5)$




We choose the group $P$ generated by

$$(1\,7\,27\,19\,18\,26\,14\,24\,30\,13\,16\,28\,31\,25\,4\,10\,22\,23\,21\,5\,8\,2\,11\,17\,15\,20\,29\,3\,9\,12\,6). \quad (30)$$

The normalizer of $P$ in the full symmetric group is the holomorph of $P$, i.e. the semidirect product of $P$ with its automorphism group. This normalizer is not contained in $A$ but $|N_{S_{31}}(P) \cap A| = 3 \times 31$. This intersection has 10 right cosets in $N_{S_{31}}(P)$. Representatives of these cosets are given by the powers of the element

$$g = (2\,13\,29\,8\,25\,16\,11\,26\,28\,21)(3\,22\,12\,30\,6\,18\,10\,27\,17\,7)(4\,14\,24\,19\,5\,23\,20\,15\,9\,31). \quad (31)$$

For $i = 1, \ldots, 9$ we form $\langle A, A^{g^i}\rangle$ and in each case obtain $A_{31}$. But a design which has $A_{31}$ as a group of automorphisms must be the complete design.

Thus, by the above theory, all designs obtained as solutions of the Kramer-Mesner system for $\lambda < \binom{v}{k}$ are pairwise non-isomorphic.

### *7.2. Second Proof*

The second proof of the fact that all designs are non-isomorphic is done using the intersection numbers of section 4.

In a first step, the global intersection type $\alpha_9^{(2)}(\mathcal{D})$ is used in order to distinguish between the designs. Clearly, two designs which have different intersection numbers are non-isomorphic.

Coming back to sections 5 and 6, we find

$$\begin{aligned} \alpha_9^{(2)}(\mathfrak{D}_1) &= 699,336750 \\ \alpha_9^{(2)}(\mathfrak{D}_2) &= 701,940750 \\ \alpha_9^{(2)}(\mathfrak{D}_3) &= 698,127750 \end{aligned} \tag{32}$$

for the designs with $\lambda = 100$ and

$$\begin{aligned} \alpha_9^{(2)}(\mathfrak{D}_1) &= 603,084075 \\ \alpha_9^{(2)}(\mathfrak{D}_2) &= 607,176075 \\ \alpha_9^{(2)}(\mathfrak{D}_3) &= 602,154075 \end{aligned} \tag{33}$$

for those with $\lambda = 93$. These numbers are all distinct (which is fine for our purposes!) but for the whole set of designs, there are coincidences.

For the 138 designs with $\lambda = 93$, we get 84 different values of $\alpha_9^{(2)}(\mathcal{D})$ in the range from 591,366075 to 611,268075.

The following table shows the classes of designs sorted according to the value of $\alpha_9^{(2)}(\mathcal{D})$. For each value, the indices $i$ of the designs $\mathfrak{D}_i$ are given.

$591,366075$ for $\{25\}$
$593,226075$ for $\{110\}$
$594,342075$ for $\{95\}$
$595,830075$ for $\{111\}$
$596,853075$ for $\{87\}$
$597,039075$ for $\{102\}$
$597,225075$ for $\{107\}$
$597,318075$ for $\{23,128\}$
$597,504075$ for $\{5,35\}$
$597,597075$ for $\{15,46\}$
$597,969075$ for $\{8\}$
$598,248075$ for $\{126\}$
$598,341075$ for $\{14\}$
$598,434075$ for $\{118,132\}$
$598,527075$ for $\{96\}$
$598,806075$ for $\{79\}$
$598,899075$ for $\{30,70,112\}$
$598,992075$ for $\{97,100\}$
$599,085075$ for $\{48\}$
$599,643075$ for $\{49\}$
$599,829075$ for $\{44,119\}$
$599,922075$ for $\{64\}$
$600,015075$ for $\{41\}$
$600,108075$ for $\{43\}$
$600,201075$ for $\{16,122\}$
$600,480075$ for $\{75\}$
$600,573075$ for $\{89\}$
$600,666075$ for $\{134\}$
$600,759075$ for $\{7,36,68\}$
$600,852075$ for $\{106,131\}$
$601,131075$ for $\{101,103,105\}$
$601,224075$ for $\{10\}$
$601,317075$ for $\{40\}$
$601,503075$ for $\{127\}$
$601,689075$ for $\{39,120,137\}$
$601,782075$ for $\{69\}$
$601,968075$ for $\{62,88\}$
$602,154075$ for $\{3,18,45,94\}$
$602,433075$ for $\{13,109\}$
$602,526075$ for $\{66\}$
$602,619075$ for $\{93,133\}$
$602,712075$ for $\{34\}$
$602,805075$ for $\{56,57,67\}$
$602,898075$ for $\{50,61,90\}$
$602,991075$ for $\{51,86\}$
$603,084075$ for $\{1,21,54,77,108,113\}$
$603,177075$ for $\{72\}$
$603,270075$ for $\{12,60,124\}$
$603,363075$ for $\{81,117\}$
$603,456075$ for $\{84\}$

$603,549075$ for $\{65\}$
$603,642075$ for $\{91,115,116\}$
$603,735075$ for $\{104\}$
$604,014075$ for $\{22\}$
$604,107075$ for $\{26\}$
$604,386075$ for $\{11,76,129\}$
$604,479075$ for $\{31,55\}$
$604,665075$ for $\{9,136\}$
$604,758075$ for $\{29,37\}$
$604,944075$ for $\{98\}$
$605,223075$ for $\{47\}$
$605,316075$ for $\{32\}$
$605,409075$ for $\{28\}$
$605,595075$ for $\{33,73,123\}$
$605,967075$ for $\{80\}$
$606,153075$ for $\{19,92\}$
$606,339075$ for $\{52,82,85,135\}$
$606,525075$ for $\{125\}$
$606,618075$ for $\{17,121\}$
$606,897075$ for $\{59\}$
$607,176075$ for $\{2,130\}$
$607,641075$ for $\{4,6\}$
$608,199075$ for $\{114\}$
$608,292075$ for $\{38,74\}$
$608,385075$ for $\{99\}$
$608,478075$ for $\{53\}$
$608,571075$ for $\{83\}$
$608,664075$ for $\{24,42\}$
$609,315075$ for $\{20\}$
$609,873075$ for $\{27\}$
$610,152075$ for $\{78\}$
$610,803075$ for $\{58\}$
$611,175075$ for $\{63\}$
$611,268075$ for $\{71,138\}$

Let us make some statistics first: The class sizes are distributed in the following way:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| # of classes of size $i$ | 48 | 23 | 10 | 2 | 0 | 1 |

The average class size is $\mu = 1.643$, we get for the variance $Var = 0.86$ and the standard deviation $\sigma = 0.93$.

A better choice for an invariant is the multiset of all block intersection types of a design. So, one starts with equation (10) and collects equal terms of $\alpha_9(B_h)$ (in the case of the 8-designs). This leads to an additive decomposition of $\alpha_9^{(2)}(\mathcal{D})$ which is a finer invariant.

For example, the class of designs with $\alpha_9^{(2)}(\mathfrak{D}_i) = 603,084075$ for $i \in \{1,21,54,77,108,113\}$ (and $\lambda = 93$) has the following *different* types of block intersections (sorted lexicographically by the coefficients of the terms):

$\mathfrak{D}_{77}$: $603,084075 = \frac{1}{2}155 \times (3 \times 10 + 100 \times 54 + 600 \times 67 + 2400 \times 68 + 3800 \times 69 + 13500 \times 70 + 8400 \times 71 + 19200 \times 73 + 10200 \times 74 + 4800 \times 75 + 18600 \times 76 + 11400 \times 77 + 7200 \times 78 + 1200 \times 79 + 3000 \times 80 + 600 \times 82 + 150 \times 88 + 30 \times 90)$

$\mathfrak{D}_{54}$: $603,084075 = \frac{1}{2}155 \times (3 \times 10 + 200 \times 57 + 1800 \times 64 + 600 \times 67 + 6000 \times 68 + 400 \times 69 + 4200 \times 70 + 4800 \times 71 + 15000 \times 72 + 10800 \times 73 + 16200 \times 74 + 13200 \times 75 + 7200 \times 76 + 10200 \times 77 + 5500 \times 78 + 4800 \times 79 + 1350 \times 80 + 2000 \times 81 + 900 \times 82 + 30 \times 90)$

$\mathfrak{D}_{1}$: $603,084075 = \frac{1}{2}155 \times (3 \times 10 + 100 \times 60 + 1400 \times 66 + 3000 \times 68 + 400 \times 69 + 1200 \times 70 + 11400 \times 71 + 22800 \times 72 + 4800 \times 73 + 17400 \times 74 + 15800 \times 75 + 7200 \times 76 + 6000 \times 77 + 12600 \times 79 + 150 \times 80 + 600 \times 82 + 300 \times 84 + 30 \times 90)$

$\mathfrak{D}_{21}$: $603,084075 = \frac{1}{2}155 \times (3 \times 10 + 400 \times 63 + 1200 \times 64 + 600 \times 67 + 2400 \times 68 + 3600 \times 69 + 6000 \times 70 + 6000 \times 71 + 12950 \times 72 + 8400 \times 73 + 14400 \times 74 + 11400 \times 75 + 17100 \times 76 + 12000 \times 77 + 5600 \times 78 + 1200 \times 79 + 600 \times 80 + 700 \times 84 + 600 \times 85 + 30 \times 90)$

$\mathfrak{D}_{113}$: $603,084075 = \frac{1}{2}155 \times (30 \times 30 + 100 \times 48 + 150 \times 60 + 600 \times 65 + 600 \times 66 + 3600 \times 69 + 9000 \times 70 + 6000 \times 71 + 9600 \times 72 + 16800 \times 73 + 12000 \times 74 + 15000 \times$

$75 + 8700 \times 76 + 12000 \times 77 + 5000 \times 78 + 4800 \times 79 + 1200 \times 82 + 3 \times 210)$
$\mathfrak{D}_{108}$: $603,084075 = \frac{1}{2}155 \times (30 \times 30 + 100 \times 48 + 600 \times 64 + 1800 \times 65 + 800 \times 66 + 900 \times 68 + 3600 \times 69 + 6000 \times 70 + 3600 \times 71 + 13600 \times 72 + 10800 \times 73 + 13200 \times 74 + 17000 \times 75 + 15750 \times 76 + 6000 \times 77 + 6600 \times 78 + 2400 \times 79 + 2400 \times 83 + 3 \times 210)$

As a matter of fact, all designs (for $\lambda = 100$ and $\lambda = 93$) can be distinguished using this invariant.

The major drawback with using the $\alpha_9^{(3)}(\mathcal{D})$ for classification purposes is simple: these numbers are quite hard to compute because lots of intersections are involved.

For sake of completeness, we list the orbit indices of the remaining 5 designs ($\mathfrak{D}_1$ has already been shown):
$\mathfrak{D}_{21}$: 1, 2, 5, 8, 10, 12, 13, 14, 18, 19, 24, 25, 28, 31, 32, 33, 36, 39, 42, 45, 46, 50, 51, 55, 60, 61, 64, 66, 68, 72, 75, 79, 84, 85, 86, 90, 96, 98, 100, 105, 107, 108, 109, 111, 114, 117, 120, 127, 128, 130, 131, 133, 134, 135, 137, 138, 141, 142, 144, 149, 151, 152, 154, 159, 167, 169, 170, 172.
$\mathfrak{D}_{54}$: 1, 2, 5, 7, 9, 12, 13, 14, 21, 22, 24, 29, 33, 34, 36, 38, 39, 42, 46, 47, 50, 53, 55, 56, 58, 60, 64, 66, 71, 72, 75, 76, 77, 81, 90, 92, 94, 95, 98, 100, 101, 102, 103, 104, 107, 109, 113, 115, 118, 119, 121, 125, 126, 128, 134, 137, 141, 145, 148, 149, 150, 151, 152, 153, 159, 163, 165, 167, 169, 171, 172.
$\mathfrak{D}_{77}$: 1, 2, 7, 8, 12, 13, 14, 18, 20, 21, 22, 24, 29, 33, 38, 39, 42, 43, 45, 46, 47, 51, 52, 55, 56, 58, 59, 64, 67, 68, 72, 78, 80, 82, 83, 86, 91, 93, 96, 100, 101, 107, 112, 113, 114, 115, 116, 117, 119, 120, 124, 125, 126, 132, 135, 136, 147, 149, 152, 153, 156, 159, 162, 165, 167, 170, 171, 172.
$\mathfrak{D}_{108}$: 1, 2, 3, 5, 9, 12, 13, 14, 20, 23, 24, 26, 32, 33, 34, 36, 38, 41, 42, 51, 52, 53, 55, 58, 60, 61, 62, 64, 67, 70, 72, 78, 79, 81, 83, 85, 86, 88, 92, 97, 98, 100, 103, 107, 111, 115, 117, 118, 119, 120, 121, 127, 131, 133, 134, 137, 139, 143, 149, 150, 151, 152, 155, 159, 162, 165, 167, 170, 172.
$\mathfrak{D}_{113}$: 1, 2, 3, 6, 11, 12, 13, 14, 17, 23, 26, 28, 30, 35, 36, 37, 42, 43, 45, 49, 55, 58, 60, 64, 66, 67, 71, 72, 73, 76, 77, 80, 82, 83, 87, 88, 100, 101, 104, 105, 109, 112, 117, 120, 121, 122, 127, 128, 130, 133, 137, 140, 141, 144, 147, 148, 149, 150, 151, 152, 154, 155, 156, 159, 165, 167, 169, 172.

In the case of the 1658 designs of type 8-$(31, 10, 100)$ we get 219 different values $\alpha_9^{(2)}(\mathcal{D})$ in the range from $688,455750$ to $716,169750$. The distribution of class sizes is the following:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| # of classes of size $i$ | 38 | 20 | 15 | 15 | 14 | 13 | 6 | 8 | 11 | 7 |
| # of classes of size $10+i$ | 11 | 14 | 9 | 11 | 5 | 3 | 4 | 5 | 2 | 2 |
| # of classes of size $20+i$ | 3 | 1 | 1 | 0 | 1 | | | | | |

The average class size is $\mu = 7.57$. We have $Var = 33.6$ and $\sigma = 5.8$.

The largest class of designs is of size 25. Namely, one gets $\alpha_9^{(2)}(\mathfrak{D}_i) = 701,661750$ for $i \in \{42, 56, 87, 116, 118, 185, 263, 356, 503, 682, 729, 737, 809, 817, 826, 1085, 1127, 1208, 1288, 1299, 1426, 1459, 1507, 1545, 1585\}$.

As remarked above, in all cases the use of block intersection numbers allowes to distinguish between the designs.

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