SOME CONNECTIONS
BETWEEN SELF-DUAL CODES,
COMBINATORIAL DESIGNS
AND SECRET-SHARING SCHEMES

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INTRODUCTION

Secret sharing is an important topic in cryptography and has applications in information security.

The age-old way to share a secret, such as the 3-digit combination 17-14-92 (combination with 100 positions) is to give part of the secret to each user: 17 to Andrew, 14 to Bryan, and 92 to Chris.

Shamir (1979) and Blakely (1979) – \((S, T)\) threshold schemes for secret sharing

A secret is transformed into a list of \(S\) shares in such manner that:

(P1) knowledge of any \(T\) shares reveals the secret, but

(P2) knowledge of \(T-1\) or fewer shares gives no information whatsoever about the secret.

McEliece and Sarwate (1981) – a formulation of \((S, T)\) threshold schemes in terms of \(q\)-ary MDS codes of block length \(n = S + 1\) with \(q^k\) codewords.
SECRET-SHARING

Secret sharing scheme – sharing a secret among a finite set of people or entities such that only some distinguished subsets of these have access to the secret.

Example:

\[ S = \{ s_1, s_2, \ldots, s_n \}, \quad U = \{ u_1, u_2, \ldots, u_p \}, \quad p > n \]

\[ U^* = \{ u_{s_1}, u_{s_2}, \ldots, u_{s_n} \} \]

Access structure – the collection of all such distinguished subsets that have access to the secret.
ACCESS STRUCTURE

If $\mathcal{P}$ is the set of parties involved in the secret-sharing, then

$$\Gamma = \{ A \subset \mathcal{P} : A \text{ can uncover the secret} \}$$

$A \in \Gamma$ - **minimum access group** if

$$B \in \Gamma \text{ and } B \subseteq A \text{ implies } B = A$$

$$\bar{\Gamma} = \{ A \mid A \text{ is a minimum access group} \}$$

$\bar{\Gamma}$ - the **minimum access structure**.

In general, determining the minimum access structure is a difficult problem.
BINARY LINEAR CODES

$GF(2)$ – a field with 2 elements.

**Binary linear** $[n, k]$ **code** $C$ **of length** $n$ – $k$-dimensional linear subspace of $GF(2)^n$.

**Weight** of a codeword $c \in C$ ($\text{wt}(c)$) – the number of nonzero components of $c$.

**Minimum weight (distance):**
$d = d(C) = \min\{\text{wt}(c) | c \in C, c \neq 0\} \rightarrow [n, k, d]$ code.

**Generator matrix of** $C$ – $k \times n$ matrix with entries in $GF(2)$ whose rows are a basis of $C$.

**Weight enumerator** of $C$: $C(y) = \sum_{i=0}^{n} A_i y^i$
SELF-DUAL CODES

• Inner product – $x.y = \sum_{i=1}^{n} x_i y_i$, $x, y \in GF(2)^n$

• Dual code – $C^\perp = \{x \in GF(2)^n \mid x.c = 0, \forall c \in C\}$

• $C$ – self-orthogonal code if $C \subseteq C^\perp$

• $C$ – self-dual code if $C = C^\perp$ ($k = n/2$)

• All codewords in a binary self-orthogonal code have even weights

• Doubly-even code – all its weights are divisible by 4

• Singly-even self-dual code – contains a codeword of weight $w \equiv 2 \pmod{4}$
EXTREMAL SELF-DUAL CODES

If $C$ is a binary self-dual $[n, n/2, d]$ code then

$$d \leq 4[n/24] + 4$$

except when $n \equiv 22 \pmod{24}$ when

$$d \leq 4[n/24] + 6$$

When $n$ is a multiple of 24, any code meeting the bound must be doubly-even.
THE SHADOW OF A SINGLY EVEN CODE

$C$ - singly even self-dual $[n, k = n/2, d]$ code

$C_0$ - its doubly even subcode:

$$C_0 = \{v \in C \mid wt(v) \equiv 0 \pmod{4}\}$$

$$\dim C_0 = k - 1$$

$$C_2 = \{v \in C \mid wt(v) \equiv 2 \pmod{4}\}$$

$$C = C_0 \cup C_2$$

$$\Rightarrow C_0^\perp = C_0 \cup C_1 \cup C_2 \cup C_3$$

$$S = C_0^\perp \setminus C = C_1 \cup C_3$$ - the shadow of $C$
A $t - (v, k, \lambda)$ design is:

- a set of $v$ points $\mathcal{P}$;
- a family of blocks $\mathcal{B} = \{ B \subset \mathcal{P}, |B| = k \}$;
- an incidence relation between them such that $v = |\mathcal{P}|$, every block is incident with precisely $k$ points, and every $t$ distinct points are incident with $\lambda$ blocks.

Any $t$-design is also a $s - (v, k, \lambda_s)$ design for $s \leq t$:

$$\lambda_s = \frac{(v - s)}{(k - s)} \lambda_{s+1} (s = 1, \ldots, t - 1), \; \lambda_t = \lambda$$
**Assmus-Mattson Theorem**

Binary case:

- $C - [n, k, d]$ binary linear code;
- $C^\perp$ – its orthogonal $[n, n - k, d^\perp]$ code;
- $t$ – an integer, $0 < t < d$, such that $C^\perp$ has not more than $d - t$ nonzero weights $w \leq n - t$.

Then:

- the supports of all codewords in $C$ of weight $u$ form a $t$-design;
- the supports of all codewords in $C^\perp$ of weight $w$, $d^\perp \leq w \leq n - t$, form a $t$-design.
SECRET-SHARING \((n - 1 \text{ PARTIES})\)

- \(s \in GF(q)\) - the secret;
- \(G = (G_0 G_1 \ldots G_{n-1})\) - a generator matrix of a code \(C\) of length \(n\);
- \(z \in GF(q)^k\) - the information vector, \(zG_0 = s\);
- \(u = zG\);
- to each party we assign \(u_i, i = 1, \ldots, n - 1\);

A scheme is said to be \textit{perfect} if a group of shares either determines the secret or gives no information about the secret.
COMPUTING THE SECRET

$s$ is determined by the set of shares \( \{u_{i_1}, u_{i_2}, \ldots, u_{i_m}\} \)

\[
\iff \quad G_0 = \sum_{j=1}^{m} x_j G_{i_j}, \quad 1 \leq i_1 < \cdots < i_m \leq n - 1
\]

\[
\iff \exists (1, 0, \ldots, 0, c_{i_1}, 0, \ldots, 0, c_{i_m}, 0 \ldots, 0) \in C^\perp, \quad (c_{i_1}, \ldots, c_{i_m}) \neq 0
\]

So by solving this linear equation, we find \( x_j \) and from then on the secret by

\[
\begin{align*}
    s &= zG_0 = \sum_{j=1}^{m} x_j zG_{i_j} = \sum_{j=1}^{m} x_j u_{i_j} \\
\end{align*}
\]
Secret-sharing based on an SD code

Dougherty, Mesnager, Sole, 2008

$D_i$ - the 1-design formed from the vectors of weight $i$

$\Gamma = \{A \mid A$ is the support of a vector $v \in C$ with $v_0 = 1\}$. 

- Any group of size less than $d - 1$ cannot recover the secret.
- There are $\lambda_1(D_i)$ groups of size $i - 1$ that can recover the secret.
- It is perfect, which means that a group of shares either determines the secret or gives no information about the secret.
- When the parties come together $\lfloor \frac{d-1}{2} \rfloor$ cheaters can be found.
Secret-sharing based on an SD code

Bouyuklieva, Varbanov, 2009

$C$ - a singly-even SD $[n, n/2, d]$ code with $wt(S) = 1$. Then, the vectors in $C_2$ (up to equivalence) are in the form $(1, c_1, c_2, \ldots, c_{n-1})$.

- Any group of size less than $d - 1$ cannot recover the secret.
- There are $A_i$ groups of size $i - 1$ that can recover the secret ($i \equiv 2 \ (mod\ 4)$).
- It is perfect, which means that a group of shares either determines the secret or gives no information about the secret.
- When the parties come together $\left\lfloor \frac{d-1}{2} \right\rfloor$ cheaters can be found.
TWO-PART SECRET SHARING

Bouyuklieva, Varbanov, 2009

• \( s' \in GF(q) \) - the second part of the secret;
• \( z \in GF(q)^k, s' = s + zG_1 + zG_2 = z(G_0 + G_1 + G_2) \);
• \( u = zG \), to each party we assign \( u_i, i = 1, \ldots, n - 1 \);

\( s' \) is determined by the set of shares \( \{u_{i_3}, u_{i_4}, \ldots, u_{i_m}\} \)

\[
\iff G_2 = G_0 + G_1 + \sum_{j=3}^{m} x_j G_{i_j}, \ 1 \leq i_1 < \cdots < i_m \leq n - 1
\]

\[
\iff \exists(1, 1, 1, 0, \ldots, 0, c_{i_3}, 0, \ldots, 0, c_{i_m}, 0, \ldots, 0) \in C^\perp, \ (c_{i_3}, \ldots, c_{i_m}) \neq 0
\]
TWO-PART SECRET SHARING

Bouyuklieva, Varbanov, 2009

Let $C$ be a binary singly-even SD $[n, n/2, d]$ code with the properties:

- $wt(S) = 1$;

- the set of codewords of weight $i$ in $C_0$ without the common zero coordinate holds a 2-design.

Then the access structure of the two parts are:

$\Gamma_1 = \{ A \mid A \text{ is the support of a vector } v \in C_2 \ (v_0 = 1) \}.$

$\Gamma_2 = \{ A \mid A \text{ is the support of a vector } v \in C_2 \\
\text{ with } v_0 = v_1 = v_2 = 1 \}.$
TWO-PART SECRET SHARING

Let $C$ be a binary doubly-even SD $[n, n/2, d]$ code with the property that the set of codewords of weight $i$ in $C$ holds a $3-(v, k, \lambda)$ design where $v = n$ and $k = i$. Then the access structure of the two parts are:

$$\Gamma_1 = \{ A \mid A \text{ is the support of a vector } v \in C \text{ with } v_0 = 1 \}.$$  

$$\Gamma_2 = \{ A \mid A \text{ is the support of a vector } v \in C \text{ with } v_0 = v_1 = v_2 = 1 \}.$$  

Let’s mention, that if a group of participants can recover the second part, to recover then the first part they need the participants 1 and 2, in general. But there are groups which can recover only the first part but not the second.
RESULTS (ONE-PART)

Let $C$ be singly-even SD code with parameters:

- $[24m + 18, 12m + 9, 4m + 4]$ or
- $[24m + 10, 12m + 5, 4m + 2]$ or
- $[24m + 2, 12m + 1, 4m + 2]$

In these cases there exist codes with $wt(S) = 1$.

\[ \Gamma = \{ A \mid A \text{ is the support of a vector } v \in C_2 \}. \]

**Example:** $C$ is $[42, 21, 8]$ code with $wt(S) = 1$ and weight enumerator
\[ 1 + 164y^8 + 697y^{10} + \ldots + 164y^{34} + y^{42}. \]

The access structure contains 164 groups of size 7, 697 groups of size 9, etc.
RESULTS (TWO-PART)

Let $C$ be singly-even SD $[24m + 2, 12m + 1, 4m + 2]$ code with $wt(S) = 1$. In this case the set of codewords of weight $i$ in $C_2$ (without the common all-one coordinate) holds a 2-design.

Example: $C$ – singly-even SD $[50, 25, 10]$ code with $wt(S) = 1$ and weight enumerator $1 + 196y^{10} + 11368y^{12} + \ldots + y^{50}$.

- For the first part of the secret, the access structure contains 196 groups of size 9.
- For the second part we take these 36 blocks of $D$ that have 1 in the first position. Without the first point, the blocks of $D$ hold $1 - (48, 8, 6)$ design $D_1$.
- We take these 6 blocks of $D_1$ that have 1 in the first position. Then, for the second part of the secret, the access structure consists of 6 groups of size 7.
RESULTS (TWO-PART)

- To recover the two-part secret should first be used the groups of size 7. They recover the second part of the secret.

- After that to recover the other part of the secret we use these groups (they are of size 8 already) and the other 30 groups of size 8. We add a new participant that has ones in these 36 groups (the other entries are 0).

- At last, we use the obtained 36 groups of size 9, and the other 160 groups of size 9 to recover the first part of the secret.
RESULTS (TWO-PART)

Let $C$ be doubly-even extremal SD $[24m + 8, 12m + 4, 4m + 4]$ code. In this case the set of codewords of weight $i$ in $C$ holds a 3-design.

**Example:** $C$ – doubly-even extremal SD $[32, 16, 8]$ code with weight enumerator $1 + 620y^8 + 13888y^{12} + \ldots + y^{32}$. The set of the codewords of weight 8 holds $3 - (32, 8, 7)$ design $D$.

- There are 155 blocks with 1 in the first position. Then, for the first part of the secret, the access structure contains 155 groups of size 7. These blocks without first point hold $2 - (31, 7, 7)$ design $D'$.

- For the second part we take these 35 blocks of $D'$ that have 1 in the first position. Without the first point, these blocks hold $1 - (30, 6, 7)$ design $D''$.

- We take these 7 blocks of $D''$ that have 1 in the first position. Then, for the second part of the secret, the access structure consists of 7 groups of size 5.
RESULTS (TWO-PART)

• To recover the two-part secret should first be used the groups of size 5. They recover the second part of the secret.

• After that to recover the other part of the secret we use these groups (they are of size 6 already) and the other 28 groups of size 6. We add a new participant that has ones in these 35 groups (the other entries are 0).

• At last, we use the obtained 35 groups of size 7, and the other 120 groups of size 7 to recover the first part of the secret.
References


THANKS FOR YOUR ATTENTION!