An algebraic approach to subsets in association schemes from finite buildings

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Outline

- Algebraic background: which techniques do we need? (Philippe Delsarte)
- Buildings: what are these structures? (Jacques Tits)
- Buildings of type F_4: some results!

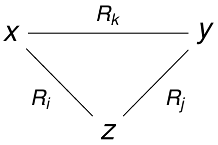
Definition of association scheme

A d-class association scheme is a pair $(\Omega, \{R_0, \ldots, R_d\})$ with Ω a set and $R_0, \ldots, R_d$ symmetric relations on Ω s.t.:

(i) $R_0 =$ identity relation,

(ii) $\{R_0, \ldots, R_d\}$ is a partition of $\Omega \times \Omega$,

(iii) there are *intersection numbers* p_{ij}^k:
if $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is p_{ij}^k.

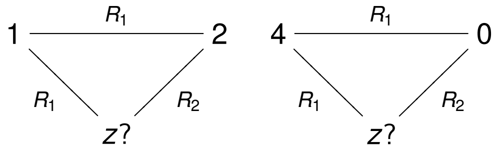


Every relation R_i is thus regular, with valency $k_i = p_{ii}^0$.

Example: the Paley scheme $P_5 = (\Omega, \{R_0, R_1, R_2\})$

- $\Omega = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.
- We define 3 relations R_0, R_1, R_2:
  - R_0 is the identity relation: e.g. $(2, 2) \in R_0$,
  - $(a, b) \in R_1$ if $a - b$ is 1 or 4 (and hence square): e.g. $(2, 3) \in R_1$,
  - $(a, b) \in R_2$ if $a - b$ is 2 or 3 (and hence non-square): e.g. $(1, 4) \in R_2$.

Intersection number p^1_{12} is 1:



z can only be 0 z can only be 3

Definition of matrices A_i

Consider an association scheme $(\Omega, \{R_0, \ldots, R_d\})$ and order the elements of Ω: $\omega_1, \ldots, \omega_{|\Omega|}$.
For each relation R_i, define the $(|\Omega| \times |\Omega|)$-matrix A_i over $\mathbb{R}$:

$$\begin{cases} (A_i)_{rs} = 1 \text{ if } (\omega_r, \omega_s) \in R_i, \\ (A_i)_{rs} = 0 \text{ if } (\omega_r, \omega_s) \notin R_i. \end{cases}$$

Properties

- A_0 is the identity matrix.
- $A_0 + \ldots + A_d$ is all-one matrix.
- A_i is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k$.

Example: Paley scheme $P_5 = (\{0,1,2,3,4\}, \{R_0, R_1, R_2\})$

$$A_0 = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array},$$

$$A_1 = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}, A_2 = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Note that A_0, A_1 and A_2 are symmetric and add up to the all-one matrix!

Decomposition into strata for scheme $(\Omega, \{R_0, \ldots, R_d\})$

$\mathbb{R}\Omega$ is the vector space with basis indexed by elements of Ω.
$\mathbb{R}\Omega$ uniquely decomposes as:

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \ldots \perp V_d,$$

with every $v \in V_i$ an eigenvector of every relation R_j: $A_j v = \lambda_{ij} v$.
These V_i are the strata, and by convention $V_0 = \langle \chi_\Omega \rangle$.

Matrix of eigenvalues P

$$P = \begin{array}{c|cccc} & R_0 & R_1 & \ldots & R_d \\ \hline V_0 & 1 & \lambda_{01} & \ldots & \lambda_{0d} \\ V_1 & 1 & \ddots & \ddots & \lambda_{1d} \\ \vdots & 1 & \ddots & \ddots & \vdots \\ V_d & 1 & \lambda_{d1} & \ldots & \lambda_{dd} \end{array}$$

Counting algebraically

The characteristic vector χ_S of $S \subseteq \Omega$ has :
$(\chi_S)_i = 1$ if $\omega_i \in S$ and $(\chi_S)_i = 0$ if $\omega_i \notin S$. So $\chi_S = (0, 1, 0, 1, 1, \dots)$.
For any $S, S' \subseteq \Omega$: $\langle \chi_S, \chi_{S'} \rangle = (1, 1, \dots, 1)(0, 1, \dots, 0) = |S \cap S'|$.

Design-orthogonality of subsets S and S'

If $\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$ and:

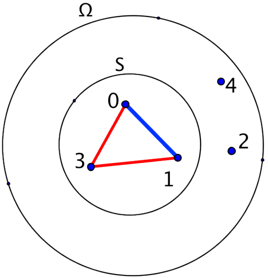
$$\begin{aligned} \chi_S &= \frac{|S|}{|\Omega|}\chi_\Omega + v_1 + \cancel{v_2} + \dots + v_d, v_i \in V_i \\ \chi_{S'} &= \frac{|S'|}{|\Omega|}\chi_\Omega + \cancel{v_1'} + v_2' + \dots + \cancel{v_d'}, v_i' \in V_i \end{aligned}$$

with $\forall i \in \{1, \dots, d\}$: $v_i = 0$ **or** $v_i' = 0$ then

$$|S \cap S'| = \langle \chi_S, \chi_{S'} \rangle = \frac{|S||S'|}{|\Omega||\Omega|}\langle \chi_\Omega, \chi_\Omega \rangle = \frac{|S||S'|}{|\Omega|}.$$

Which components v_i of χ_S are zero?

The inner distribution $\mathbf{a}$ of $S \subseteq \Omega$: $\mathbf{a}_i = \frac{1}{|S|}|(S \times S) \cap R_i|$.
So $\mathbf{a}_i$ is average valency of R_i in S,and $\mathbf{a}_0 = 1, \mathbf{a}_0 + \ldots + \mathbf{a}_d = |S|$.
Example: Paley scheme $(\Omega, \{R_0, R_1, R_2\})$, and $S = \{0, 1, 3\}$.



$$\mathbf{a} = \frac{1}{3}\Big((1, 1, 1)_0 + (1, 1, 1)_1 + (1, 0, 2)_3\Big) = (1, \tfrac{2}{3}, \tfrac{4}{3}).$$

Which components v_i of χ_S are zero?

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \ldots \perp V_d$$

$\chi_S = \frac{|S|}{|\Omega|}\chi_\Omega + v_1 + \ldots + v_d$, with $v_i \in V_i$.
For any (non-empty) subset $S \subseteq \Omega$, inner distribution **a** satisfies:

$$(\mathbf{a}P^{-1})_i \geq 0,$$

with equality if and only if $v_i = 0$ (or hence $\chi_S \in (V_i)^\perp$).

The Johnson scheme $J(n,k)$ in a set X of size n

- Ω: the $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots1}$ k-subsets in X,
- $(\omega_1, \omega_2) \in R_i \iff |\omega_1 \cap \omega_2| = k - i$.

The Grassmann scheme $J_q(n,k)$ in $V(n,q)$

- Ω: the $\left[{n \atop k} \right]_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}$ k-subspaces in $V(n,q)$,
- $(\omega_1, \omega_2) \in R_i \iff dim(\omega_1 \cap \omega_2) = k - i$.

Delsarte's characterization in $J(n,k)$ and $J_q(n,k)$

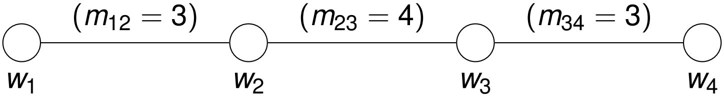
- In both schemes: $\mathbb{R}\Omega = V_0 \perp V_1 \perp \ldots \perp V_d$ with $d = \min(k, n-k)$ (with "natural ordering").
- In $J(n,k)$: $S \subseteq \Omega$ is $t-(n,k,\lambda)$-design (i.e. every t-subset in X is in λ elements of S) $\iff \chi_S \in V_0 \perp \cancel{V_1} \perp \ldots \perp \cancel{V_t} \perp V_{t+1} \perp \ldots \perp V_d$.
- In $J_q(n,k)$: $S \subseteq \Omega$ is $t-(n,k,\lambda;q)$-design (i.e. every t-space in $V(n,q)$ is in λ elements of S) $\iff \chi_S \in V_0 \perp \cancel{V_1} \perp \ldots \perp \cancel{V_t} \perp V_{t+1} \perp \ldots \perp V_d$.

Where else do we have these characterizations in schemes and their q-analogues?

Buildings of rank n are certain incidence structures with n types (Tits). Each finite building has a Coxeter-Dynkin diagram, for instance:

- Each node corresponds with a type (points, lines,planes,...).
- If $x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_n$ of types $1, \ldots, i-1, i+1, \ldots, n$ are incident,
  they are incident with $q_i + 1$ common objects of type i.
- The thin building with $(q_1, \ldots, q_n) = (1, \ldots, 1)$ is constructed from “Coxeter group" itself,
  thick building is more complicated but similar.

Coxeter group W and its thin building



- W is generated by involutions w_i, with $(w_i w_j)^{m_{ij}} = e$ (we draw no line if $m_{ij} = 2 \iff w_i w_j = w_j w_i$),
- maximal parabolic subgroup $P_i := \langle \{w_1, \ldots, w_n\} \backslash \{w_i\} \rangle$,
- objects of type i: cosets wP_i,
- aP_i and bP_j are incident iff $aP_i \cap bP_j \neq \emptyset$,
- (aP_i, bP_j) and $(a'P_i, b'P_j)$ in same relation iff $P_i a^{-1} bP_j = P_i a'^{-1} b' P_j$ (if $i = j$, this is an association scheme in SOME cases!)

$A_n \cong$ symmetric group of size $(n+1)!$ on $n+1$ elements.

Thin buildings of type A_n, with $q_1 = \ldots = q_n = 1 \rightarrow$ Johnson scheme!

- objects of type i: the $\binom{n+1}{i}$ subsets of size i,
- 2 objects of type i are k-related iff size intersection is $i-k$.

Thick buildings of type A_n, with $q_1 = \ldots = q_n = q \rightarrow$ Grassmann scheme!

- objects of type i: the $\begin{bmatrix} n+1 \\ i \end{bmatrix}_q$ i-spaces of $V(n+1, q)$,
- two objects of type i are k-related iff dim intersection is $i-k$.

$B_n \cong$ hyperoctahedral group of size $2^n n!$

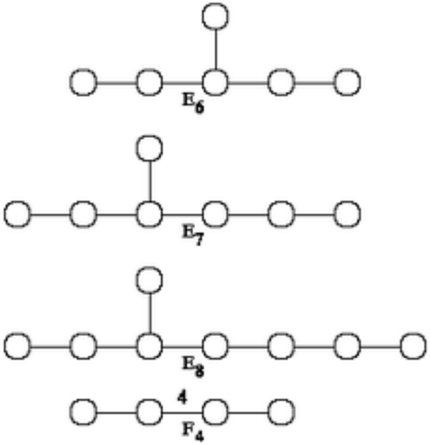
Thin building, with $q_1 = \ldots = q_n = 1$->coding theory!

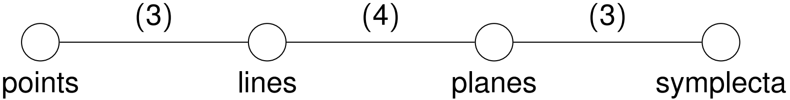
- type n: 2^n binary codewords of length n,
- 2 objects of type n are k-related iff they are equal in $n - k$ positions.

Thick building, with $q_1 = \ldots = q_{n-1} = q, q_n = q^e$->polar spaces!

- type n: $(q^e + 1) \ldots (q^{n-1+e} + 1)$ maximals of rank n polar space,
- 2 maximals are k-related iff dim intersection is $n - k$.

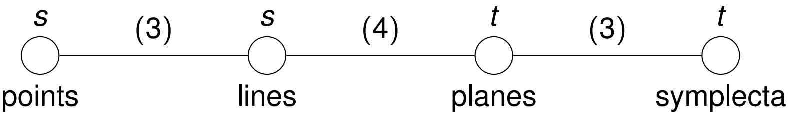
What about other finite Coxeter groups?





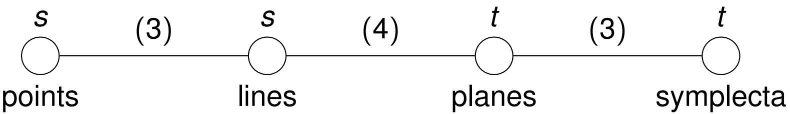
Thin building of type F_4

- $|F_4| = 1152 = 2^7.3^2$,
- 24 points, 96 lines and planes, 24 symplecta.
- Points: all 24 vectors in $\mathbb{R}^4$ of form $(\pm 1, \pm 1, 0, 0)$ up to permutation.
- Relations between points: R_0, R_1, R_2, R_3, R_4 correspond with inner products $2, 1, 0, -1, -2$ (24-cell).



Thick building of type F_4

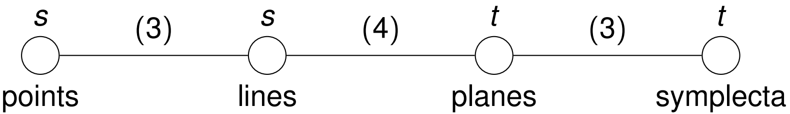
- $(s, t) = (q, 1), (1, q), (q, q), (q, q^2)$ or (q^2, q),
- number of points: $|\Omega| = (s^2 + s + 1)(s^2 t + 1)(s^2 t^2 + 1)(s^3 t^3 + 1)$,
- association scheme on points:
  R_0: identity,
  R_1: on unique common line (collinear),
  R_2: in unique common symplecton (cohyperlinear),
  R_3: unique common neighbour wrt collinearity (almost opposite),
  R_4: no common neighbour (opposite).



Matrix of eigenvalues P of association scheme on points (Gomi):

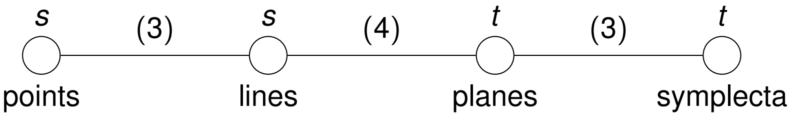
$\mathbb{R}\Omega$ decomposes into strata: $V_0 \perp V_1 \perp V_2 \perp V_3 \perp V_4$.

	R_0	R_1	R_2	R_3	R_4
V_0	1	$s(s+1)(st+1)\left(st^2+1\right)$	$s^4t\left(t^2+t+1\right)\left(st^2+1\right)$	$s^5t^3(s+1)(st+1)\left(st^2+1\right)$	s^9t^6
V_1	1	$(s+1)(st+1)(st+s-1)$	$s^2\left(t^2+t+1\right)\left(s^2t-1\right)$	$s^3t(s+1)(st+1)(st-t-1)$	$-s^6t^3$
V_2	1	$\left(-1+s^2\right)(st+1)$	$\left(s^3t-s^2t^2-s^2t-st-s+t\right)s$	$-(s-1)(s+1)(st+1)s^2t$	s^5t^2
V_3	1	$-1-st-st^2+s^2$	$s(t-s)\left(t^2+t+1\right)$	$st\left(s^2t^2-t^2+st+s\right)$	$-s^3t^3$
V_4	1	$-(st+1)(st^2+1)$	$st\left(t^2+t+1\right)\left(st^2+1\right)$	$-(st+1)\left(st^2+1\right)st^3$	t^6s^3



The set of points S inside an object

- points in plane interact as in projective plane (so all collinear):
  inner distribution of S: $\mathbf{a} = (1, s^2 + s, 0, 0, 0)$,
  $\mathbf{a}P^{-1} = (?, ?, ?, ?, 0) \Longrightarrow \chi_S \in V_0 \perp V_1 \perp V_2 \perp V_3 \perp \cancel{V_4}$.
- points in symplecton interact as in rank 3 polar space (collinear or cohyperlinear):
  inner distribution of S: $\mathbf{a} = (1, s(s+1)(st+1), s^4 t, 0, 0)$.
  $\mathbf{a}P^{-1} = (?, ?, ?, 0, 0) \Longrightarrow \chi_S \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$.



Notion of design T in F_4-building

- Set of points S in any plane : $\chi_S \in V_0 \perp V_1 \perp V_2 \perp V_3 \perp \cancel{V_4}$,
  So if $\chi_T \in V_0 \perp \cancel{V_1} \perp \cancel{V_2} \perp \cancel{V_3} \perp V_4$
  $\implies |S \cap T| = \frac{|S||T|}{|\Omega|} = |S|/((s^2t+1)(s^2t^2+1)(s^3t^3+1))$.
- Set of points S in any symplecton: $\chi_S \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$,
  So if $\chi_T \in V_0 \perp \cancel{V_1} \perp \cancel{V_2} \perp V_3 \perp V_4$
  $\implies |S \cap T| = \frac{|S||T|}{|\Omega|} = |S|/((s^2t^2+1)(s^3t^3+1))$.

Cliques of the oppositeness-relation R_4

If S is a set of mutually opposite points (hence not collinear with a common point):

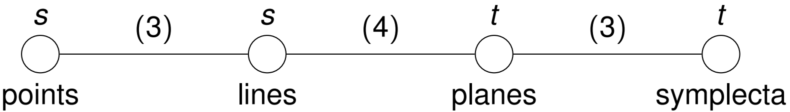
$$\mathbf{a} = (1, 0, 0, 0, |S| - 1).$$

$\mathbb{R}\Omega = V_0 \perp V_1 \perp V_2 \perp V_3 \perp V_4$, matrix of eigenvalues P:

	R_0	R_1	R_2	R_3	R_4
V_0	1	$s(s+1)(st+1)\left(st^2+1\right)$	$s^4 t\left(t^2+t+1\right)\left(st^2+1\right)$	$s^5 t^3 (s+1)(st+1)\left(st^2+1\right)$	$s^9 t^6$
V_1	1	$(s+1)(st+1)(st+s-1)$	$s^2\left(t^2+t+1\right)\left(s^2 t-1\right)$	$s^3 t (s+1)(st+1)(st-t-1)$	$-s^6 t^3$
V_2	1	$\left(-1+s^2\right)(st+1)$	$\left(s^3 t - s^2 t^2 - s^2 t - st - s + t\right) s$	$-(s-1)(s+1)(st+1) s^2 t$	$s^5 t^2$
V_3	1	$-1 - st - st^2 + s^2$	$s(t-s)\left(t^2+t+1\right)$	$st\left(s^2 t^2 - t^2 + st + s\right)$	$-s^3 t^3$
V_4	1	$-(st+1)(st^2+1)$	$st\left(t^2+t+1\right)\left(st^2+1\right)$	$-(st+1)\left(st^2+1\right) st^3$	$t^6 s^3$

- $(\mathbf{a}P^{-1})_i \geq 0$ with equality iff $\chi_S \in (V_i)^\perp$.
- $i = 1$ yields best bound: $|S| \leq s^3 t^3 + 1$, with equality iff $\chi_S \in V_0 \perp \cancel{V_1} \perp V_2 \perp V_3 \perp V_4$.

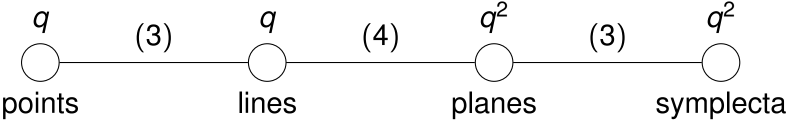
Embedding of one F_4-building in another



Embed points T of F_4-building with parameters (s', t') in this building:
(inner distribution $\mathbf{a}$ of T consists just of valencies in smallest building)

- if $(s', t') = (q, 1), (s, t) = (q, q)$, then
  $\chi_T \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$,
- if $(s', t') = (q, q), (s, t) = (q, q^2)$, then
  $\chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp \cancel{V_3} \perp \cancel{V_4}$,
- if $(s', t') = (q, q), (s, t) = (q^2, q)$, then
  $\chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp V_3 \perp \cancel{V_4}$,

An immediate consequence of the previous for $(s,t) = (q,q^2)$!



- S: set of $1 + s^3t^3 = 1 + q^9$ mutually opposite points
  $\Longrightarrow \chi_S \in V_0 \perp \cancel{V_1} \perp V_2 \perp V_3 \perp V_4$,
- T: point set of embedded F_4-building with $(s',t') = (q,q)$:
  $\Longrightarrow \chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp \cancel{V_3} \perp \cancel{V_4}$.

Design-orthogonality yields: $|S \cap T| = \langle \chi_S, \chi_T \rangle = \frac{|S||T|}{|\Omega|} = q^3 + 1$.

Thank you for your attention!