ARCS IN PROJECTIVE GEOMETRIES OVER $\mathbb{F}_4$ AND QUATERNARY LINEAR CODES

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1. Preliminaries

$\mathbb{F}_q$, $q = p^r$, p – prime, the field with q elements

Definition. A **multiset** in $\mathrm{PG}(k-1,q)$ is a mapping

$$\mathcal{K} : \left\{ \begin{array}{lcl} \mathcal{P} & \rightarrow & \mathbb{N}_0, \\ P & \rightarrow & \mathcal{K}(P). \end{array} \right.$$

$\mathcal{K}(P)$ – the **multiplicity** of the point P.

$\mathcal{Q} \subset \mathcal{P}$: $\mathcal{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{K}(P)$.

$\mathcal{K}(\mathcal{P})$ – the **cardinality** of $\mathcal{K}$.

Points, lines, ... ,hyperplanes of multiplicity i are called i-points, i-lines, ... , i-hyperplanes.

a_i – the number of i-hyperplanes

$(a_i)_{i\geq 0}$ – the **spectrum** of $\mathcal{K}$

Definition. **(n, w)-arc** in $\mathrm{PG}(k-1, q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. **(n, w)-blocking set with respect to hyperplanes** in $\mathrm{PG}(k-1, q)$ (or **(n, w)-minihyper**): a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \geq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

2. Linear codes over finite fields

C – linear $[n,k,d]_q$ code:

C is a linear subspace of $\mathbb{F}_q^n$ with $\dim C = k$,

$\delta_{\mathsf{Ham}}(u,v) \geq d$ for every $u,v \in C$, $u \neq v$.

Problem A. For given k, d and q find the smallest value of n for which there exists an $[n,k,d]_q$-code.

The **Griesmer** bound:

$$n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$$

$\diamond$ For k, q fixed, there exist codes meeting the Griesmer bound for all sufficiently large d. (**Tamari**)

$\diamond$ For d, q fixed, $k \to \infty$ Griesmer codes do not exist. (**Dodunekov**)

Hence it is reasonable to attack the problem for "small" fields $\mathbb{F}_q$ and "small" dimensions k.

At present:

$q = 2$,	$k \leq 8$ – the problem is solved for all d;
$q = 3$,	$k \leq 5$ – the problem is solved for all d;
$q = 4$,	$k \leq 4$ – the problem is solved for all d;
	$k = 5$, ≈ 110 **open cases**;
$q = 5$,	$k \leq 3$ – the problem is solved for all d;
	$k = 4$ – only **4** open cases for d:
	$d = 81, 82, 161, 162$;
$q = 7, 8, 9$	$k \leq 3$ the problem is solved for all d.

3. Arcs and linear codes

Theorem. The existence of an $[n, k, d]_q$-code of full length is equivalent to that of an $(n, n-d)$-arc in $\mathrm{PG}(k-1, q)$.

◇ C – $[n, k, d]_q$-code with $n = t + g_q(k, d)$;

◇ $\mathcal{K}$ - $(n, n-d)$-arc associated with C;

◇ $\gamma_i :=$ maximal multiplicity of an i-dimensional subspace of $\mathrm{PG}(k-1, q)$, $i = 0, 1, \ldots, k-1$,

$$\gamma_i \leq t + g_q(i+1, d).$$

Problem B. Characterize geometrically all Griesmer codes with given parameters k, d and q. Equivalently: Characterize all minihypers in $\mathrm{PG}(k-1, q)$ with parameters

$$\left(\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i,\right), 0 \leq \epsilon_i \leq q-1,$$

where $v_i = (q^i - 1)/(q-1)$.

- probably hopeless in all generality
- Belov, Logachev, Sandimirov, 1974
- N. Hamada,T. Helleseth
- L. Storme, J. De Beule, P. Govaerts et al.
- A. Klein, Kl. Metsch and many others

3.1. Divisibility of arcs in $\mathrm{PG}(k-1, q)$

Definition. A (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ is called **divisible** if there exists an integer $\Delta > 1$ such that $\mathcal{K}(H) \equiv n \pmod{\Delta}$ for every hyperplane H.

Theorem. (**H. N. Ward**) Let $\mathcal{K}$ be a Griesmer (n, w)-arc in $\mathrm{PG}(k-1, p)$, p a prime, with $w \equiv n \pmod{p^e}$, $e \geq 1$. Then $\mathcal{K}(H) \equiv n \pmod{p^e}$ for any hyperplane H.

Theorem. (**H. N. Ward**) Let $\mathcal{K}$ be a Griesmer (n, w)-arc in $\mathrm{PG}(k-1, 4)$ with $w \equiv n \pmod{2^e}$, $e \geq 1$. Then $\mathcal{K}(H) \equiv n \pmod{2^{e-1}}$ for any hyperplane H.

3.2. Extension of arcs in $\mathrm{PG}(k-1,q)$

Definition. An (n,w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called **extendable** if there exists an $(n+1,w)$-arc $\mathcal{K}^*$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}^*(P) \ge \mathcal{K}(P)$ for every point $P \in \mathcal{P}$.

Theorem. (**Hill**, **Lizak**) Let $\mathcal{K}$ be an (n,w)-arc in $\mathrm{PG}(k-1,q)$ with $\gcd(n-w,q)=1$. Let further $\mathcal{K}(H) \equiv n$ or $w \pmod q$ for all hyperplanes H. Then $\mathcal{K}$ is extendable to an $(n+1,w)$-arc in $\mathrm{PG}(k-1,q)$.

4. The status quo for $q = 4$

Problem.

For codes over $\mathbb{F}_4$, $n_4(k, d)$ has been found for $k \leq 4$ for all d.

For $k = 5$, $n_4(5, d)$ has been found for all but ≈110 values of d.

Some open cases for $k = 5$, $q = 4$:

d	$g_4(5, d)$	$n_4(5, d)$	(n, w)-arc $\mathcal{K}$	$\mathcal{K}\|_H$
333	446	446–447	$(446, 113)$-arc	$(113, 29)$-arc in $\mathrm{PG}(3, 4)$
334	447	447–448	$(447, 113)$-arc	
335	448	448–449	$(448, 113)$-arc	
336	449	449–450	$(449, 113)$-arc	

d	$g_4(5,d)$	$n_4(5,d)$	(n,w)-arc $\mathcal{K}$	$\mathcal{K}\|_H$
345	462	462–463	$(462,117)$-arc	$(117,30)$-arc in $\mathrm{PG}(3,4)$
346	463	463–464	$(463,117)$-arc	
347	464	464–465	$(464,117)$-arc	
348	465	465–466	$(465,117)$-arc	
349	467	467–468	$(467,118)$-arc	$(118,30)$-arc in $\mathrm{PG}(3,4)$
350	468	468–469	$(468,118)$-arc	
351	469	469–470	$(469,118)$-arc	
352	470	470–471	$(470,118)$-arc	

5. Characterization of the $(118, 30)$-arcs in $\mathrm{PG}(3, 4)$

Theorem. Let $\mathcal{K}$ be a $(118, 30)$-arc. Then

$$\gamma_0 = 2, \quad \gamma_1 = 8, \quad \gamma_2 = 30.$$

Moreover, the possible multiplicities of hyperplanes are 14, 18, 22, 26, 30.

5.1. Constructions using a $(128,32)$-arc in $\mathrm{PG}(3,4)$

Step 1. ℓ – a line in $\mathrm{PG}(3,4)$;

$\pi_0,\ldots,\pi_4$ – the planes through ℓ

$$\mathcal{L}(P)=\begin{cases} 0 & \text{if } P\in\ell,\\ 1 & \text{if } P\in(\pi_0\cup\pi_1)\setminus\ell,\\ 2 & \text{if } P\in(\pi_2\cup\pi_3\cup\pi_4)\setminus\ell,\end{cases}$$

Step 2. Delete a $(10,2)$-minihyper $\mathcal{F}$ with $\mathcal{F}(P)\le\mathcal{L}(P)$ for every point P.

Possibilities for $\mathcal{F}$:

(a) two skew lines different from ℓ;

(b) two intersecting lines different from ℓ; the common point is not on π_0 or π_1.

$$\mathcal{K} = \mathcal{L} - \mathcal{F}$$

is a $(118, 30)$-arc in $\mathrm{PG}(3, 4)$ with one of the following spectra:

(a) $a_{14} = 2$, $a_{22} = 0$, $a_{26} = 10$, $a_{30} = 73$,
$\lambda_0 = 9$, $\lambda_1 = 34$, $\lambda_2 = 42$;

(b) $a_{14} = 2$, $a_{22} = 1$, $a_{26} = 8$, $a_{30} = 74$,
$\lambda_0 = 10$, $\lambda_1 = 32$, $\lambda_2 = 43$.

6.2. Constructions of arcs with $a_{18} \neq 0$

Step 1. ℓ – a line in $\mathrm{PG}(3,4)$;

$\pi_0, \ldots, \pi_4$ – the planes through ℓ

$$\mathcal{L}(P) = \begin{cases} 1 & \text{if } P \in (\pi_0 \cup \pi_1), \\ 2 & \text{if } P \in (\pi_2 \cup \pi_3 \cup \pi_4) \setminus \ell, \end{cases}$$

Step 2. Delete a $(15,3)$-minihyper $\mathcal{F}$ contained in $\pi_2 \cup \pi_3 \cup \pi_4$ and meeting ℓ in exactly three points.

Possibilities for $\mathcal{F}$:

(a) three skew lines contained in π_2, π_3 and π_4, respectively;

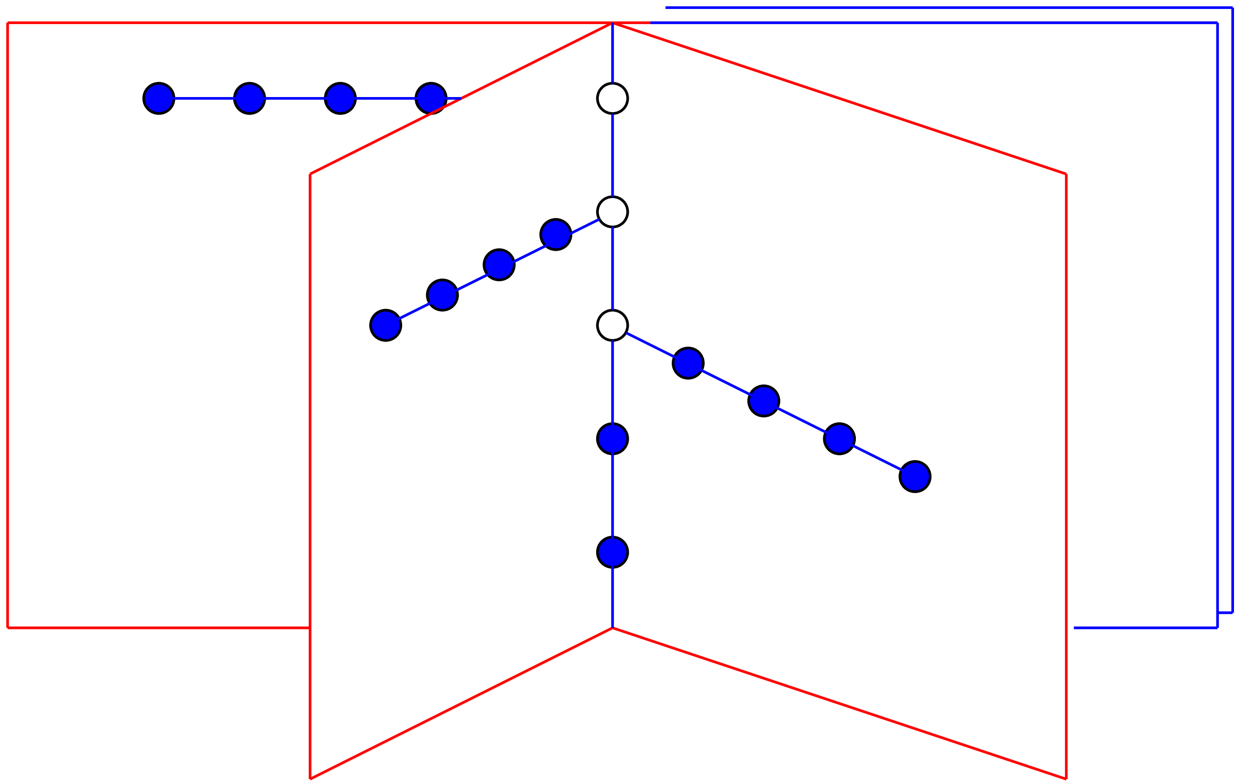
(b) $\mathrm{PG}(3,2)$ constructed in $\pi_2 \cup \pi_3 \cup \pi_4$ and meeting ℓ in three points.

$\mathcal{K} = \mathcal{L} - \mathcal{F}$ is a $(118, 30)$-arc in $\mathrm{PG}(3,4)$ with spectrum:

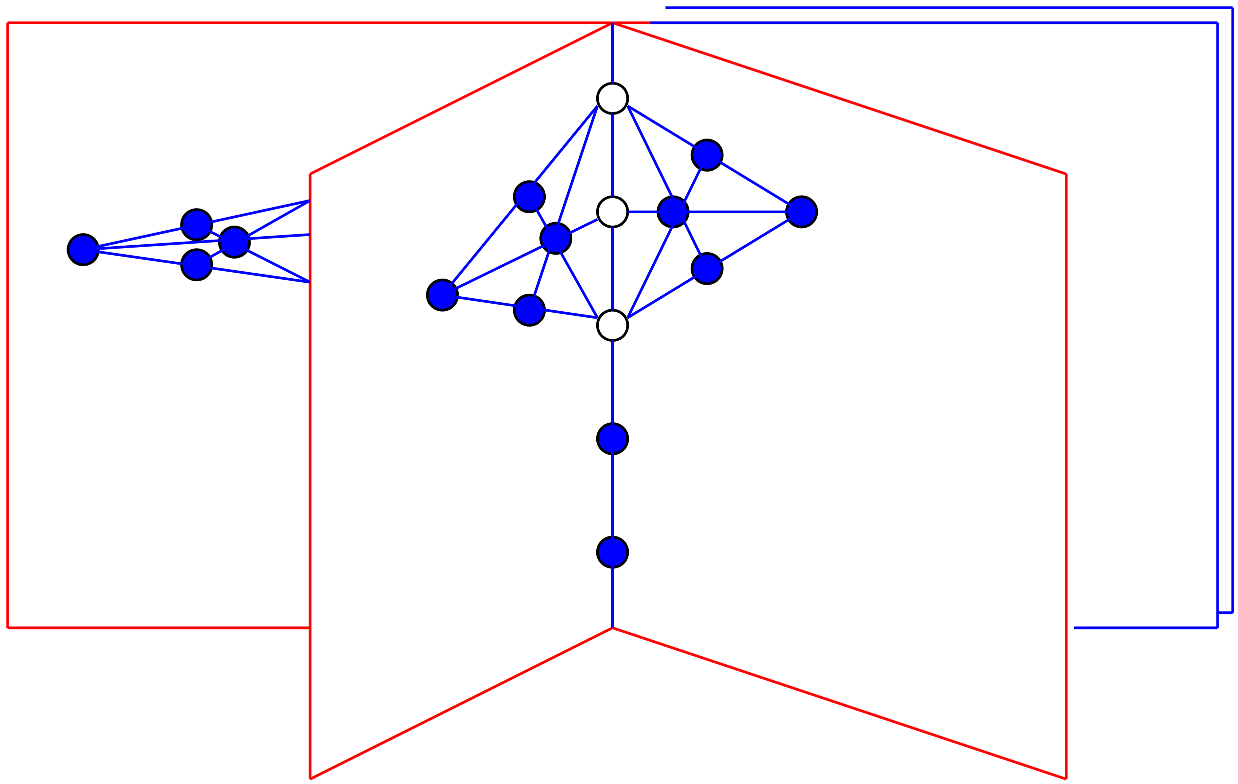
$a_{18} = 2$, $a_{26} = 12$, $a_{30} = 71$,

$\lambda_0 = 3$, $\lambda_1 = 46$, $\lambda_2 = 36$.

(a)



(b)



6.3. Arcs with weights 22,26,30 – dual constructions

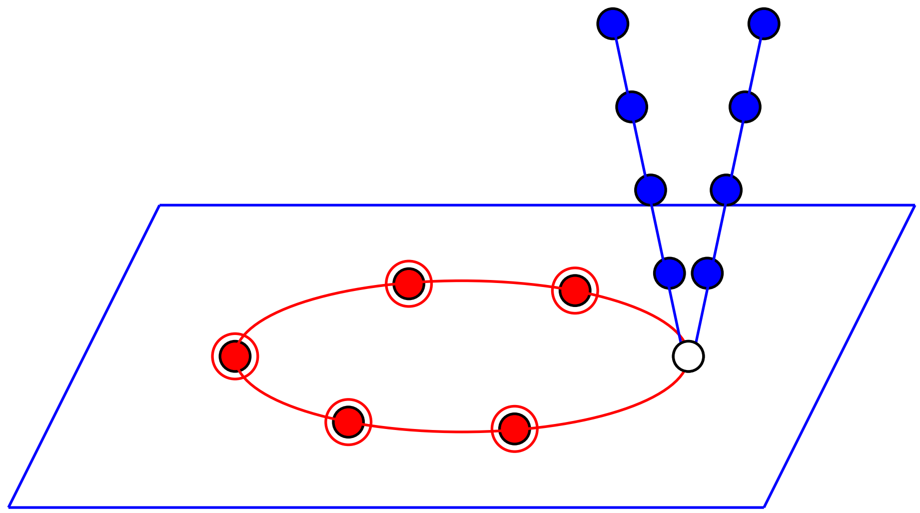
Theorem. There exists a one-to-one correspondence between the arcs $\mathcal{K}$ with parameters $(118, \{22, 26, 30\})$ and the arcs $\widetilde{\mathcal{K}}$ with parameters $(18, \{2, 6, 10\})$ in $\mathrm{PG}(3, 4)$.

Proof.

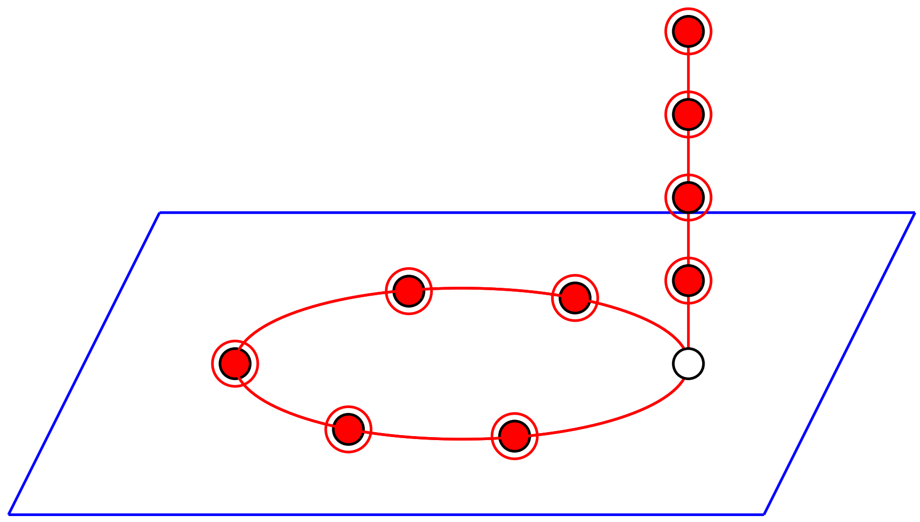
30-planes	$\longrightarrow$	0-points
26-planes	$\longrightarrow$	1-points
22-planes	$\longrightarrow$	2-points

0-points	$\longrightarrow$	10-planes
1-points	$\longrightarrow$	6-planes
2-points	$\longrightarrow$	2-planes

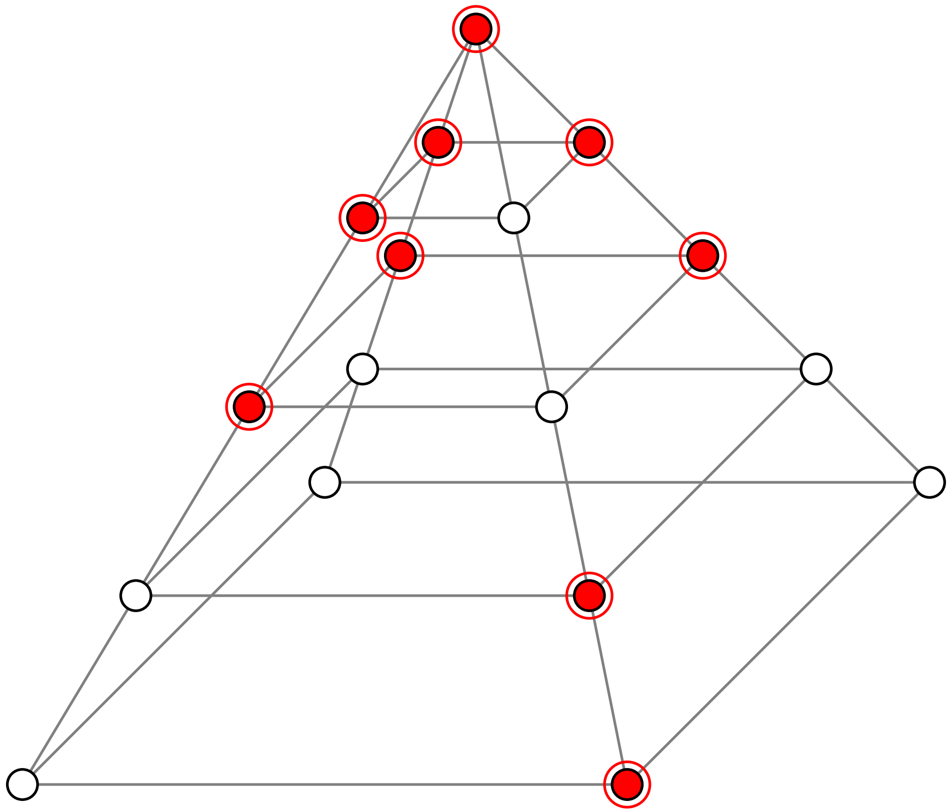
(c) $(18, \{2, 6, 10\})$-arc



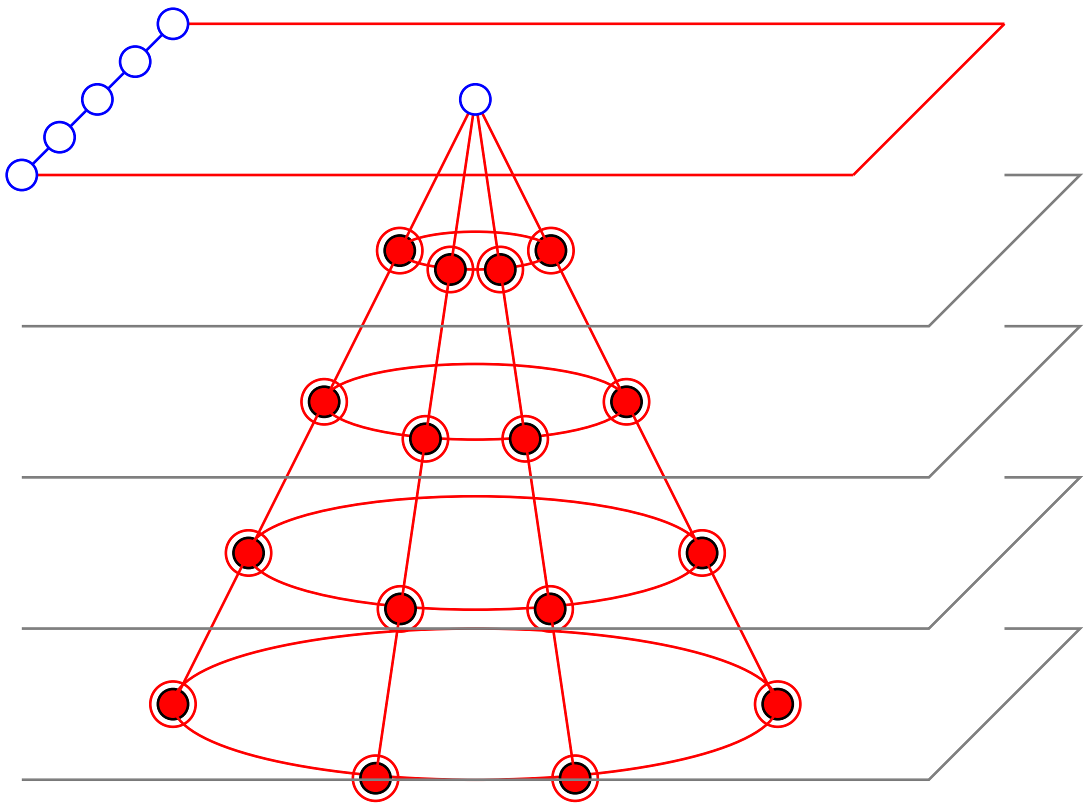
(d) $(18, \{2, 6, 10\})$-arc



(e) $(18, \{2, 6, 10\})$-arc



(d) $(118, 30)$-arc



6. Characterization of the $(117, 30)$-arcs in $\mathrm{PG}(3, 4)$

Theorem. Every $(117, 30)$-arc in $\mathrm{PG}(3, 4)$ is extendable to a $(118, 30)$-arc.

Proof.

Let $\mathcal{K}$ be a $(117, 30)$-arc in $\mathrm{PG}(3, 4)$.

The possible multiplicities of hyperplanes are all $\equiv 1$ and $2 \pmod 4$.

Hence $\mathcal{K}$ is extendable by Hill and Lizak's Extension Theorem.

7. Characterization of the $(113, 29)$-arcs in $\mathrm{PG}(3, 4)$

Theorem. Let $\mathcal{K}$ be a $(113, 29)$-arc. Then

$$\gamma_0 = 2, \quad \gamma_1 = 8, \quad \gamma_2 = 29.$$

Moreover, the possible multiplicities of hyperplanes are 13, 17, 21, 25, 27, 29.

We can get a $(113, 29)$-arc by deleting a line from a $(118, 30)$-arc.

Apart from this we have the following possibilities:

7.1. Constructions using a $(128, 32)$-arc in $\mathrm{PG}(3, 4)$

Step 1. ℓ – a line in $\mathrm{PG}(3, 4)$;

$\pi_0, \ldots, \pi_4$ – the planes through ℓ

$$\mathcal{L}(P) = \begin{cases} 0 & \text{if } P \in \ell, \\ 1 & \text{if } P \in (\pi_0 \cup \pi_1) \setminus \ell, \\ 2 & \text{if } P \in (\pi_2 \cup \pi_3 \cup \pi_4) \setminus \ell, \end{cases}$$

Step 2. Delete a $(15, 3)$-minihyper $\mathcal{F}$ with $\mathcal{F}(P) \leq \mathcal{L}(P)$ for every point P.

$\mathcal{F}$ is the complement of a plane hyperoval.

$\mathcal{K} = \mathcal{L} - \mathcal{F}$ – a $(113, 29)$-arc in $\mathrm{PG}(3,4)$ **with** $\mathcal{K}(H) \equiv 1 \pmod 4$ for all H.

7.2. Construction using a $(28,8)$-arc in $\mathrm{PG}(3,4)$

$\mathcal{L}$ – a $(28,8)$-arc in $\mathrm{PG}(3,4)$;

Spectrum: $a_0 = 1, a_4 = 21, a_8 = 63$

$\mathcal{K} = 1 + \mathcal{L}$ is a $(113,29)$-arc in $\mathrm{PG}(3,4)$;

2-points – the points of $\mathcal{L}$;

1-points – all other points.

$\mathcal{K}(H) \equiv 1 \pmod 4$ for all planes H.

7.3. Constructions using a $(49, 13)$-arc in $\mathrm{PG}(3, 4)$

$\mathcal{L}$ – a $(49, 13)$-arc in $\mathrm{PG}(3, 4)$;

Spectrum: $a_1 = 1, a_9 = 16, a_{13} = 68$

π – a fixed plane in $\mathrm{PG}(3, 4)$;

$\mathcal{A}$ – a $(64, 16)$-arc in $\mathrm{PG}(3, 4)$:

$$\mathcal{A}(P) = \begin{cases} 0 & \text{if } P \in \pi, \\ 1 & \text{if } P \notin \pi. \end{cases}$$

$\mathcal{K} = \mathcal{L} + \mathcal{A}$ – a $(113, 29)$-arc in $\mathrm{PG}(3, 4)$ **with** $\mathcal{K}(H) \equiv 1 \pmod 4$ for all H.

7.4. The exceptional $(113, 29)$-arc in $\mathrm{PG}(3, 4)$

ℓ – a fixed line;

$\pi_0, \ldots, \pi_4$ – the planes through ℓ;

$\mathcal{O}$ – an oval in π_0 with nucleus N, $\ell \cap (\mathcal{O} \cup \{N\}) = \emptyset$;

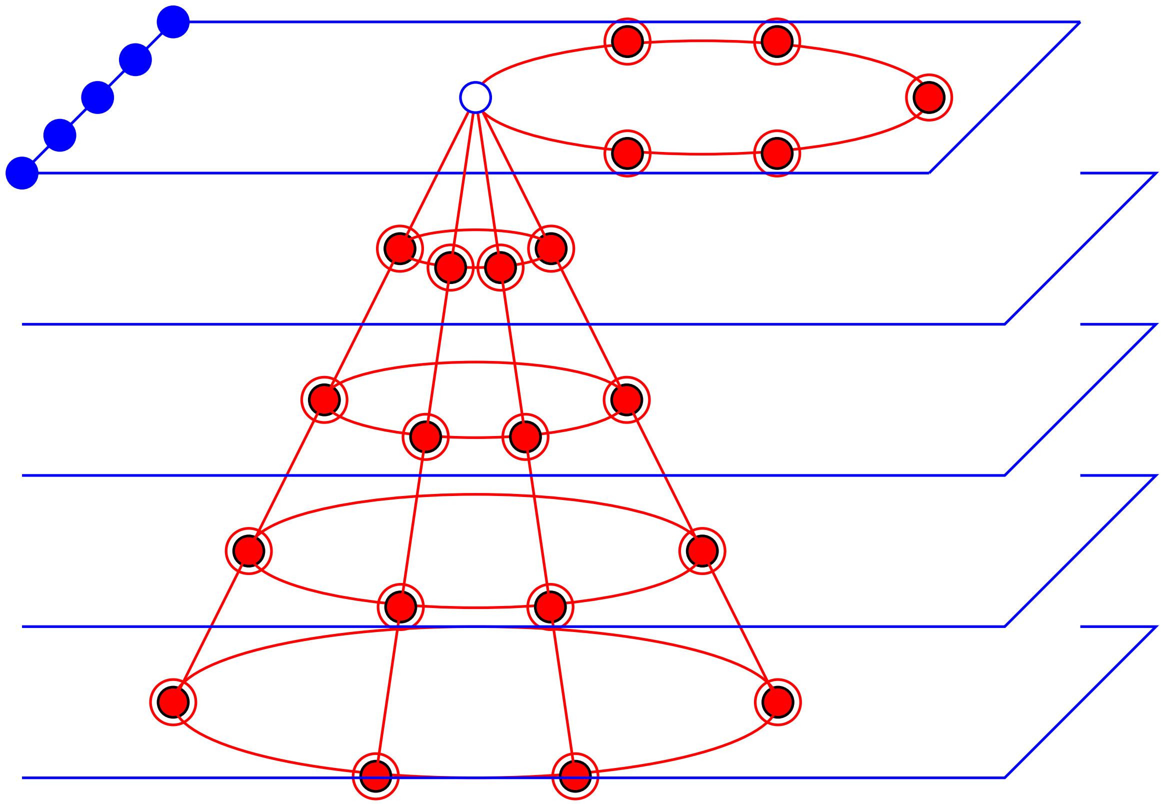
$\mathcal{H}$ – a hyperoval in π_1, $\mathcal{H} \cap \ell = \emptyset$;

C - a cone with vertex N and base curve $\mathcal{H}$

The arc $\mathcal{K}$:

- 0-point - the point N;
- 2-points - the points of the cone (without N) and the points of $\mathcal{O}$;
- 1-points - all other points.

$\mathcal{K}$ is the only $(113, 29)$-arc which has 27-planes, i.e. planes H with $\mathcal{K}(H) \not\equiv 1 \pmod 4$.



8. Applications of the characterization

Theorem. There are no $(448, 113)$-arcs in $\mathrm{PG}(4, 4)$. Equivalently, there are no $[448, 5, 336]_4$-codes and $n_4(5, 335) = 449$, $n_4(5, 336) = 450$.

Theorem. There are no $(464, 117)$-arcs in $\mathrm{PG}(4, 4)$. Equivalently, there are no $[464, 5, 347]_4$-codes and $n_4(5, 347) = 465$, $n_4(5, 348) = 466$.

Theorem. There are no $(467, 118)$-arcs in $\mathrm{PG}(4, 4)$. Equivalently, there are no $[467, 5, 349]_4$-codes and $n_4(5, 349) = 468$, $n_4(5, 350) = 469$, $n_4(5, 351) = 470$, $n_4(5, 352) = 471$.

d	$g_4(5,d)$	$n_4(5,d)$
333	446	446–447
334	447	447–448
335	448	449
336	449	450
345	462	462–463
346	463	463–464
347	464	465
348	465	466
349	467	468
350	468	469
351	469	470
352	470	471

8. Characterization of the $(102, 26)$-arcs in $\mathrm{PG}(3, 4)$

8.1. The $(101, 26)$- and $(102, 26)$-arcs in $\mathrm{PG}(3, 4)$

Theorem. There exists exactly one $(102, 26)$-arc in $\mathrm{PG}(3, 4)$. It is obtained as the sum of an ovoid and the complete space.

Spectrum:

$$a_{22} = 17, a_{26} = 68, \quad \lambda_0 = 0, \lambda_1 = 68, \lambda_2 = 17.$$

Theorem. Every $(101, 26)$-arc in $\mathrm{PG}(3, 4)$ is extendable to a $(102, 26)$-arc.

d	$g_4(5,d)$	$n_4(5,d)$	(n,w)-arc $\mathcal{K}$	$\mathcal{K}\|_H$
297	398	399	$(399,101)$-arc	$(101,26)$-arc in $\mathrm{PG}(3,4)$
298	399	400	$(400,101)$-arc	
299	400	401	$(401,101)$-arc	
300	401	402	$(402,101)$-arc	
301	403	404	$(404,102)$-arc	$(102,26)$-arc in $\mathrm{PG}(3,4)$
302	404	405	$(405,102)$-arc	
303	405	406	$(406,102)$-arc	
304	406	407	$(407,102)$-arc	

Open problem. Characterize geometrically the arcs with parameters

$$(q^3 + 2q^2 + q + 2, q^2 + 2q + 2) \text{ in } \mathrm{PG}(3, q), q > 2.$$

These arcs are associated with Griesmer codes with parameters

$$[q^3 + 2q^2 + q + 2, 4, q^3 + q^2 - q]_q.$$

An obvious construction: the sum of an ovoid and the whole space $\mathrm{PG}(3, q)$.

The question is: are there other constructions?

- **In $\mathrm{PG}(3,3)$:** We have two $(50,17)$-arcs:

(a) the sum of a cap and the whole space;

(b) two copies of $\mathrm{PG}(3,3)$ minus two different planes π_0, π_1 minus a line (skew to the line $\ell = \pi_0 \cap \pi_1$).

- **In $\mathrm{PG}(3,4)$:** There is exacly one $(102,26)$-arc and it is the sum of an ovoid and the whole space.

- **In $\mathrm{PG}(3,5)$:** There is exacly one $(182,37)$-arc and it is the sum of an ovoid and the whole space.

Conjecture. (At least) for every prime $p \geq 5$ there is a unique arc with parameters $(p^3 + 2p^2 + p + 2, p^2 + 2p + 2)$ in $\mathrm{PG}(3, p)$. It is obtained as the sum of an ovoid and the whole space.

How can one prove this?

8.2. Reducibility of plane $(x(q+1)+1, x)$-minihypers

The planes of maximal multiplicity have parameters $(q^2+2q+2, q+3)$.

The existence of such arcs is equivalent to that of minihypers with parameters $(q^2, q-1)$ (with maximal multiplicity of a point equal to 2).

These parameters can be written as $(x(q+1)+1, x)$ with $x = q-1$.

Reducible $(x(q+1)+1, x)$-minihypers. can be obtained from $(x(q+1), x)$-minihypers by adding a point.

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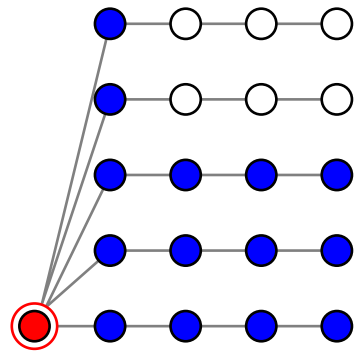
Irreducible $(x(q+1)+1, x)$-minihypers.

- the complement of an oval for all odd q
- for $q = 4$: one irreducible $(16, 3)$-minihyper
- for $q = 5$: one further irreducible minihyper with $\lambda_2 = 2, \lambda_0 = 8$.

$(16, 3)$-minihyper in $\mathrm{PG}(2, 4)$



$(25, 4)$-minihyper in $\mathrm{PG}(2, 5)$