ARCS IN PROJECTIVE GEOMETRIES
OVER $\mathbb{F}_4$ AND QUATERNARY LINEAR CODES

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1. Preliminaries

$\mathbb{F}_q$, $q = p^r$, $p$ - prime, the field with $q$ elements

**Definition.** A *multiset* in $\text{PG}(k-1, q)$ is a mapping

$$\mathcal{K} : \begin{cases} \mathcal{P} & \rightarrow & \mathbb{N}_0, \\ P & \rightarrow & \mathcal{K}(P). \end{cases}$$

$\mathcal{K}(P)$ - the *multiplicity* of the point $P$.

$Q \subset \mathcal{P}$: $\mathcal{K}(Q) = \sum_{P \in Q} \mathcal{K}(P)$.

$\mathcal{K}(\mathcal{P})$ - the *cardinality* of $\mathcal{K}$. 
Points, lines, ..., hyperplanes of multiplicity \( i \) are called \( i \)-points, \( i \)-lines, ..., \( i \)-hyperplanes.

\( a_i \) – the number of \( i \)-hyperplanes

\( (a_i)_{i \geq 0} \) – the spectrum of \( \mathcal{K} \)
**Definition.** $(n, w)$-arc in $\text{PG}(k - 1, q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane $H$: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane $H_0$: $\mathcal{K}(H_0) = w$.

**Definition.** $(n, w)$-blocking set with respect to hyperplanes in $\text{PG}(k - 1, q)$ (or $(n, w)$-minihyper): a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane $H$: $\mathcal{K}(H) \geq w$;

3) there exists a hyperplane $H_0$: $\mathcal{K}(H_0) = w$. 
2. Linear codes over finite fields

$C$ - linear $[n, k, d]_q$ code:

$C$ is a linear subspace of $\mathbb{F}_q^n$ with $\dim C = k$, 

$\delta_{\text{Ham}}(u, v) \geq d$ for every $u, v \in C$, $u \neq v$.

**Problem A.** For given $k$, $d$ and $q$ find the smallest value of $n$ for which there exists an $[n, k, d]_q$-code.

The **Griesmer** bound:

$$n_q(k, d) \geq g_q(k, d) = \sum_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor$$
For $k, q$ fixed, there exist codes meeting the Griesmer bound for all sufficiently large $d$. (Tamari)

For $d, q$ fixed, $k \to \infty$ Griesmer codes do not exist. (Dodunekov)

Hence it is reasonable to attack the problem for “small” fields $\mathbb{F}_q$ and “small” dimensions $k$.

At present:

- $q = 2$, $k \leq 8$ – the problem is solved for all $d$;
- $q = 3$, $k \leq 5$ – the problem is solved for all $d$;
- $q = 4$, $k \leq 4$ – the problem is solved for all $d$;
  - $k = 5$, $\approx 110$ open cases;
- $q = 5$, $k \leq 3$ – the problem is solved for all $d$;
  - $k = 4$ – only 4 open cases for $d$:
    - $d = 81, 82, 161, 162$;
- $q = 7, 8, 9$, $k \leq 3$ the problem is solved for all $d$. 
3. Arcs and linear codes

**Theorem.** The existence of an $[n, k, d]_q$-code of full length is equivalent to that of an $(n, n - d)$-arc in $\text{PG}(k - 1, q)$.

$\diamond C - [n, k, d]_q$-code with $n = t + g_q(k, d)$;

$\diamond \mathcal{K} - (n, n - d)$-arc associated with $C$;

$\diamond \gamma_i :=$ maximal multiplicity of an $i$-dimensional subspace of $\text{PG}(k - 1, q)$, $i = 0, 1, \ldots, k - 1$,

$$\gamma_i \leq t + g_q(i + 1, d).$$
Problem B. Characterize geometrically all Griesmer codes with given parameters $k, d$ and $q$. Equivalently: Characterize all minihypers in $\text{PG}(k - 1, q)$ with parameters

$$\left( \sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i, \right), 0 \leq \epsilon_i \leq q - 1,$$

where $v_i = (q^i - 1)/(q - 1)$.

- probably hopeless in all generality
- Belov, Logachev, Sandimirov, 1974
- N. Hamada, T. Helleseth
- L. Storme, J. De Beule, P. Govaerts et al.
- A. Klein, Kl. Metsch and many others
3.1. Divisibility of arcs in $\text{PG}(k-1, q)$

**Definition.** A $(n, w)$-arc $\mathcal{K}$ in $\text{PG}(k-1, q)$ is called **divisible** if there exists an integer $\Delta > 1$ such that $\mathcal{K}(H) \equiv n \pmod{\Delta}$ for every hyperplane $H$.

**Theorem.** (H. N. Ward) Let $\mathcal{K}$ be a Griesmer $(n, w)$-arc in $\text{PG}(k-1, p)$, $p$ a prime, with $w \equiv n \pmod{p^e}$, $e \geq 1$. Then $\mathcal{K}(H) \equiv n \pmod{p^e}$ for any hyperplane $H$.

**Theorem.** (H. N. Ward) Let $\mathcal{K}$ be a Griesmer $(n, w)$-arc in $\text{PG}(k-1, 4)$ with $w \equiv n \pmod{2^e}$, $e \geq 1$. Then $\mathcal{K}(H) \equiv n \pmod{2^{e-1}}$ for any hyperplane $H$. 
3.2. Extension of arcs in $\text{PG}(k - 1, q)$

**Definition.** An $(n, w)$-arc $\mathcal{K}$ in $\text{PG}(k - 1, q)$ is called **extendable** if there exists an $(n + 1, w)$-arc $\mathcal{K}^*$ in $\text{PG}(k - 1, q)$ with $\mathcal{K}^*(P) \geq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$.

**Theorem. (Hill, Lizak)** Let $\mathcal{K}$ be an $(n, w)$-arc in $\text{PG}(k - 1, q)$ with $\gcd(n - w, q) = 1$. Let further $\mathcal{K}(H) \equiv n$ or $w \pmod q$ for all hyperplanes $H$. Then $\mathcal{K}$ is extendable to an $(n + 1, w)$-arc in $\text{PG}(k - 1, q)$. 
4. The status quo for $q = 4$

Problem.

For codes over $\mathbb{F}_4$, $n_4(k, d)$ has been found for $k \leq 4$ for all $d$.

For $k = 5$, $n_4(5, d)$ has been found for all but $\approx 110$ values of $d$.

Some open cases for $k = 5$, $q = 4$:

\begin{tabular}{|c|c|c|c|c|}
\hline
$d$ & $g_4(5, d)$ & $n_4(5, d)$ & $(n, w)$-arc $\mathcal{K}$ & $\mathcal{K}|_H$ \\
\hline
333 & 446 & 446–447 & (446, 113)-arc & \\
334 & 447 & 447–448 & (447, 113)-arc & (113, 29)-arc in PG(3, 4) \\
335 & 448 & 448–449 & (448, 113)-arc & \\
336 & 449 & 449–450 & (449, 113)-arc & \\
\hline
\end{tabular}
| $d$ | $g_4(5, d)$ | $n_4(5, d)$ | $(n, w)$-arc $\mathcal{K}$ | $\mathcal{K}|_H$ |
|-----|-------------|-------------|-----------------|-----------------|
| 345 | 462         | 462–463     | (462, 117)-arc  |                 |
| 346 | 463         | 463–464     | (463, 117)-arc  | (117, 30)-arc   |
| 347 | 464         | 464–465     | (464, 117)-arc  | in PG(3, 4)     |
| 348 | 465         | 465–466     | (465, 117)-arc  |                 |
| 349 | 467         | 467–468     | (467, 118)-arc  |                 |
| 350 | 468         | 468–469     | (468, 118)-arc  | (118, 30)-arc   |
| 351 | 469         | 469–470     | (469, 118)-arc  | in PG(3, 4)     |
| 352 | 470         | 470–471     | (470, 118)-arc  |                 |
5. Characterization of the \((118, 30)\)-arcs in \(PG(3, 4)\)

**Theorem.** Let \(\mathcal{K}\) be a \((118, 30)\)-arc. Then

\[\gamma_0 = 2, \quad \gamma_1 = 8, \quad \gamma_2 = 30.\]

Moreover, the possible multiplicities of hyperplanes are 14, 18, 22, 26, 30.
5.1. Constructions using a (128, 32)-arc in PG(3, 4)

Step 1. \( \ell \) – a line in PG(3, 4);

\( \pi_0, \ldots, \pi_4 \) – the planes through \( \ell \)

\[
\mathcal{L}(P) = \begin{cases} 
0 & \text{if } P \in \ell, \\
1 & \text{if } P \in (\pi_0 \cup \pi_1) \setminus \ell, \\
2 & \text{if } P \in (\pi_2 \cup \pi_3 \cup \pi_4) \setminus \ell,
\end{cases}
\]

Step 2. Delete a (10, 2)-minihyper \( \mathcal{F} \) with \( \mathcal{F}(P) \leq \mathcal{L}(P) \) for every point \( P \).

Possibilities for \( \mathcal{F} \):

(a) two skew lines different from \( \ell \);

(b) two intersecting lines different from \( \ell \); the common point is not on \( \pi_0 \) or \( \pi_1 \).
\[ \mathcal{K} = \mathcal{L} - \mathcal{F} \]

is a \((118, 30)\)-arc in \(\text{PG}(3, 4)\) with one of the following spectra:

(a) \quad a_{14} = 2, \ a_{22} = 0, \ a_{26} = 10, \ a_{30} = 73, \\
\lambda_0 = 9, \ \lambda_1 = 34, \ \lambda_2 = 42;

(b) \quad a_{14} = 2, \ a_{22} = 1, \ a_{26} = 8, \ a_{30} = 74, \\
\lambda_0 = 10, \ \lambda_1 = 32, \ \lambda_2 = 43.
6.2. Constructions of arcs with $a_{18} \neq 0$

Step 1. $\ell$ – a line in $\text{PG}(3, 4)$;

$\pi_0, \ldots, \pi_4$ – the planes through $\ell$

$$\mathcal{L}(P) = \begin{cases} 1 & \text{if } P \in (\pi_0 \cup \pi_1), \\ 2 & \text{if } P \in (\pi_2 \cup \pi_3 \cup \pi_4) \setminus \ell, \end{cases}$$

Step 2. Delete a $(15, 3)$-minihyper $\mathcal{F}$ contained in $\pi_2 \cup \pi_3 \cup \pi_4$ and meeting $\ell$ in exactly three points.
Possibilities for $\mathcal{F}$:

(a) three skew lines contained in $\pi_2$, $\pi_3$ and $\pi_4$, respectively;

(b) $\text{PG}(3, 2)$ constructed in $\pi_2 \cup \pi_3 \cup \pi_4$ and meeting $\ell$ in three points.

$\mathcal{K} = \mathcal{L} - \mathcal{F}$ is a $(118, 30)$-arc in $\text{PG}(3, 4)$ with spectrum:

$a_{18} = 2$, $a_{26} = 12$, $a_{30} = 71$,

$\lambda_0 = 3$, $\lambda_1 = 46$, $\lambda_2 = 36$. 
6.3. Arcs with weights 22, 26, 30 – dual constructions

**Theorem.** There exists a one-to-one correspondence between the arcs $\mathcal{K}$ with parameters $(118, \{22, 26, 30\})$ and the arcs $\tilde{\mathcal{K}}$ with parameters $(18, \{2, 6, 10\})$ in $PG(3, 4)$.

**Proof.**

30-planes $\rightarrow$ 0-points  
26-planes $\rightarrow$ 1-points  
22-planes $\rightarrow$ 2-points  

0-points $\rightarrow$ 10-planes  
1-points $\rightarrow$ 6-planes  
2-points $\rightarrow$ 2-planes
(c) \((18, \{2, 6, 10\})\)-arc
(d) $(18, \{2, 6, 10\})$-arc
(e) $(18, \{2, 6, 10\})$-arc
(d) (118, 30)-arc
6. Characterization of the \((117, 30)\)-arcs in \(\text{PG}(3, 4)\)

**Theorem.** Every \((117, 30)\)-arc in \(\text{PG}(3, 4)\) is extendable to a \((118, 30)\)-arc.

**Proof.**

Let \(\mathcal{K}\) be a \((117, 30)\)-arc in \(\text{PG}(3, 4)\).

The possible multiplicities of hyperplanes are all \(\equiv 1\) and \(2 \pmod{4}\).

Hence \(\mathcal{K}\) is extendable by Hill and Lizak’s Extension Theorem.
7. Characterization of the (113, 29)-arcs in PG(3, 4)

**Theorem.** Let $\mathcal{K}$ be a (113, 29)-arc. Then

$$\gamma_0 = 2, \quad \gamma_1 = 8, \quad \gamma_2 = 29.$$  

Moreover, the possible multiplicities of hyperplanes are 13, 17, 21, 25, 27, 29.

We can get a (113, 29)-arc by deleting a line from a (118, 30)-arc.

Apart from this we have the following possibilities:
7.1. Constructions using a $(128,32)$-arc in $\text{PG}(3,4)$

**Step 1.** $\ell$ – a line in $\text{PG}(3,4)$;

$\pi_0,\ldots,\pi_4$ – the planes through $\ell$

$$\mathcal{L}(P) = \begin{cases} 
0 & \text{if } P \in \ell, \\
1 & \text{if } P \in (\pi_0 \cup \pi_1) \setminus \ell, \\
2 & \text{if } P \in (\pi_2 \cup \pi_3 \cup \pi_4) \setminus \ell,
\end{cases}$$

**Step 2.** Delete a $(15,3)$-minihyper $\mathcal{F}$ with $\mathcal{F}(P) \leq \mathcal{L}(P)$ for every point $P$. 
$\mathcal{F}$ is the complement of a plane hyperoval.

$\mathcal{K} = \mathcal{L} - \mathcal{F}$ – a $(113, 29)$-arc in $\text{PG}(3, 4)$ with $\mathcal{K}(H) \equiv 1 \pmod{4}$ for all $H$. 
7.2. Construction using a \((28, 8)\)-arc in \(\text{PG}(3, 4)\)

\(\mathcal{L}\) – a \((28, 8)\)-arc in \(\text{PG}(3, 4)\);

Spectrum: \(a_0 = 1, a_4 = 21, a_8 = 63\)

\(\mathcal{K} = 1 + \mathcal{L}\) is a \((113, 29)\)-arc in \(\text{PG}(3, 4)\);

2-points – the points of \(\mathcal{L}\);

1-points – all other points.

\(\mathcal{K}(H) \equiv 1 \pmod{4}\) for all planes \(H\).
7.3. Constructions using a $(49, 13)$-arc in $\operatorname{PG}(3, 4)$

$\mathcal{L}$ – a $(49, 13)$-arc in $\operatorname{PG}(3, 4)$;

Spectrum: $a_1 = 1, a_9 = 16, a_{13} = 68$

$\pi$ – a fixed plane in $\operatorname{PG}(3, 4)$;

$\mathcal{A}$ – a $(64, 16)$-arc in $\operatorname{PG}(3, 4)$:

$$
\mathcal{A}(P) = \begin{cases} 
0 & \text{if } P \in \pi, \\
1 & \text{if } P \notin \pi.
\end{cases}
$$

$\mathcal{K} = \mathcal{L} + \mathcal{A}$ – a $(113, 29)$-arc in $\operatorname{PG}(3, 4)$ with $\mathcal{K}(H) \equiv 1 \pmod{4}$ for all $H$. 

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7.4. The exceptional $(113, 29)$-arc in $PG(3, 4)$

\( \ell \) – a fixed line;

\( \pi_0, \ldots, \pi_4 \) – the planes through \( \ell \);

\( \mathcal{O} \) – an oval in \( \pi_0 \) with nucleus \( N \), \( \ell \cap (\mathcal{O} \cup \{N\}) = \emptyset \);

\( \mathcal{H} \) – a hyperoval in \( \pi_1 \), \( \mathcal{H} \cap \ell = \emptyset \);

\( C \) – a cone with vertex \( N \) and base curve \( \mathcal{H} \)

The arc \( \mathcal{K} \):

0-point – the point \( N \);

2-points – the points of the cone (without \( N \)) and the points of \( \mathcal{O} \);

1-points – all other points.
$\mathcal{K}$ is the only $(113, 29)$-arc which has 27-planes, i.e. planes $H$ with $\mathcal{K}(H) \not\equiv 1 \pmod{4}$. 
8. Applications of the characterization

**Theorem.** There are no $(448, 113)$-arcs in $\text{PG}(4, 4)$. Equivalently, there are no $[448, 5, 336]_4$-codes and $n_4(5, 335) = 449$, $n_4(5, 336) = 450$.

**Theorem.** There are no $(464, 117)$-arcs in $\text{PG}(4, 4)$. Equivalently, there are no $[464, 5, 347]_4$-codes and $n_4(5, 347) = 465$, $n_4(5, 348) = 466$.

**Theorem.** There are no $(467, 118)$-arcs in $\text{PG}(4, 4)$. Equivalently, there are no $[467, 5, 349]_4$-codes and $n_4(5, 349) = 468$, $n_4(5, 350) = 469$, $n_4(5, 351) = 470$, $n_4(5, 352) = 471$. 
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8. Characterization of the (102, 26)-arcs in $\text{PG}(3, 4)$

8.1. The (101, 26)- and (102, 26)-arcs in $\text{PG}(3, 4)$

**Theorem.** There exists exactly one (102, 26)-arc in $\text{PG}(3, 4)$. It is obtained as the sum of an ovoid and the complete space.

Spectrum:

$$a_{22} = 17, a_{26} = 68, \quad \lambda_0 = 0, \lambda_1 = 68, \lambda_2 = 17.$$

**Theorem.** Every (101, 26)-arc in $\text{PG}(3, 4)$ is extendable to a (102, 26)-arc.
| $d$  | $g_4(5, d)$ | $n_4(5, d)$ | $(n, w)$-arc $\mathcal{K}$ | $\mathcal{K}|_H$          |
|------|------------|-------------|-----------------------------|---------------------------|
| 297  | 398        | 399         | (399, 101)-arc              |                           |
| 298  | 399        | 400         | (400, 101)-arc              | (101, 26)-arc in PG(3, 4) |
| 299  | 400        | 401         | (401, 101)-arc              |                           |
| 300  | 401        | 402         | (402, 101)-arc              |                           |
| 301  | 403        | 404         | (404, 102)-arc              |                           |
| 302  | 404        | 405         | (405, 102)-arc              | (102, 26)-arc in PG(3, 4) |
| 303  | 405        | 406         | (406, 102)-arc              |                           |
| 304  | 406        | 407         | (407, 102)-arc              |                           |
Open problem. Characterize geometrically the arcs with parameters

\[(q^3 + 2q^2 + q + 2, q^2 + 2q + 2)\] in \(\text{PG}(3, q), q > 2\).

These arcs are associated with Griesmer codes with parameters

\([q^3 + 2q^2 + q + 2, 4, q^3 + q^2 - q]_q\).

An obvious construction: the sum of an ovoid and the whole space \(\text{PG}(3, q)\).

The question is: are there other constructions?
• **In PG(3, 3):** We have two (50, 17)-arcs:

(a) the sum of a cap and the whole space;

(b) two copies of PG(3, 3) minus two different planes $\pi_0, \pi_1$ minus a line (skew to the line $\ell = \pi_0 \cap \pi_1$).

• **In PG(3, 4):** There is exactly one (102, 26)-arc and it is the sum of an ovoid and the whole space.

• **In PG(3, 5):** There is exactly one (182, 37)-arc and it is the sum of an ovoid and the whole space.
Conjecture. (At least) for every prime $p \geq 5$ there is a unique arc with parameters $(p^3 + 2p^2 + p + 2, p^2 + 2p + 2)$ in $\text{PG}(3, p)$. It is obtained as the sum of an ovoid and the whole space.

How can one prove this?
8.2. Reducibility of plane \((x(q + 1) + 1, x)\)-minihypers

The planes of maximal multiplicity have parameters \((q^2 + 2q + 2, q + 3)\).

The existence of such arcs is equivalent to that of minihypers with parameters \((q^2, q - 1)\) (with maximal multiplicity of a point equal to 2).

These parameters can be written as \((x(q + 1) + 1, x)\) with \(x = q - 1\).
Reducible \((x(q + 1) + 1, x)\)-minihypers. can be obtained from \((x(q + 1), x)\)-minihypers by adding a point.

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Irreducible \((x(q + 1) + 1, x)\)-minihypers.

- the complement of an oval for all odd \(q\)
- for \(q = 4\): one irreducible \((16, 3)\)-minihyper
- for \(q = 5\): one further irreducible minihyper with \(\lambda_2 = 2, \lambda_0 = 8\).
(16, 3)-minihyper in $\text{PG}(2, 4)$

(25, 4)-minihyper in $\text{PG}(2, 5)$