Erdős-Ko-Rado Theorems for dual polar spaces

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April 12 2010
The first Erdős-Ko-Rado Theorem

**E.K.R. [1961]**

If $\Omega$ is a set with $n$ elements and $S$ is a family of subsets of size $k$ of $\Omega$, with $n \geq 2k$, such that the elements of $S$ are pairwise intersecting, then $|S| \leq \binom{n-1}{k-1}$.

**Characterization of the families of maximum size**

If $|S| = \binom{n-1}{k-1}$, then:

- $2k < n$ and $S$ is the family of subsets of size $k$ containing a fixed element of $\Omega$.
- $2k = n$ and $S$ is either the family of subsets of size $k$ containing a fixed element of $\Omega$ or it consists of the representatives of all the complementary pairs.
Several different variants of this theorem have been proved.

**B.M.I. Rands [1982]**

The largest set of blocks of a $t - (v, k, \lambda)$ design pairwise intersecting has size equal to the number of blocks through a point and the blocks through a point is the only set of blocks meeting the bound, provided $v \geq f(k, t)$. 
If $V$ is a $n$–dimensional vector space over $\mathbb{F}_q$ and $S$ is a family of $k$–dimensional subspaces of $V$ pairwise intersecting non–trivially, with $n \geq 2k$, then $|S| \leq \left\lfloor \frac{n-1}{k-1} \right\rfloor_q$. If $|S| = \left\lfloor \frac{n-1}{k-1} \right\rfloor_q$, then:

- $2k < n$ and $S$ is the set of $k$–dimensional subspaces containing a fixed non–zero vector of $V$.
- $2k = n$ and $S$ is either the set of $k$–dimensional subspaces containing a fixed non–zero vector of $V$ or it is the set of $k$–dimensional subspaces of $V$ contained in a hyperplane.
Graph theoretic approach

Ω: set of vertices for the graph Γ (k–subsets, k–subspaces...).
Two vertices are adjacent iff their intersection is trivial.
A EKR set is a coclique of Γ.
If Γ is a ν–regular graph with least eigenvalue τ and S is a coclique of Γ, then

\[ |S| \leq \frac{|\Omega|}{1 - \frac{\nu}{\tau}} \]

and if |S| meets the bound, then its characteristic vector \( \chi_S \) is such that

\[ \chi_S = \frac{|S|}{|\Omega|} \mathbf{1} + u, \]

where \( u \) is an eigenvector with eigenvalue τ.
Classical finite polar spaces

Classical finite polar spaces are incidence structures consisting of the lattices of subspaces of a finite projective space totally isotropic with respect to a certain non-degenerate sesquilinear form.

- the parabolic quadric $Q(2n, q)$: $(n - 1)$-dimensional generators,
- the hyperbolic quadric $Q^+(2n + 1, q)$: $n$-dimensional generators,
- the elliptic quadric $Q^-(2n + 1, q)$: $(n - 1)$-dimensional generators,
- the symplectic space $W(2n + 1, q)$: $n$-dimensional generators,
- the hermitian variety $H(2n, q^2)$: $(n - 1)$-dimensional generators,
- the hermitian variety $H(2n + 1, q^2)$: $n$-dimensional generators.
The analogue problem in this setting is finding the largest size for a set of pairwise intersecting subspaces of a polar space and characterizing the sets meeting the bound. We deal with the case of generators of polar spaces, when their dimension is at least two.
## The bounds

| Stanton [1980]: | upper bound for $|S|$ | Example of set meeting the bound |
|----------------|------------------------|----------------------------------|
| Polar space    |                        |                                  |
| $Q(2n, q)$     | $\prod_{i=1}^{n-1} (q^i + 1)$ | generators through a point      |
| $Q^+(2n+1, q)$, n odd | $\prod_{i=0}^{n-1} (q^i + 1)$ | generators through a point      |
| $Q^+(2n+1, q)$, n even | $\prod_{i=1}^{n} (q^i + 1)$ | generators of one family        |
| $Q^-(2n+1, q)$ | $\prod_{i=2}^{n} (q^i + 1)$ | generators through a point      |
| $W(2n+1, q)$   | $\prod_{i=1}^{n} (q^i + 1)$ | generators through a point      |
| $H(2n, q^2)$   | $\prod_{i=1}^{n-1} (q^{2i+1} + 1)$ | generators through a point      |
| $H(2n+1, q^2)$, n odd | $\prod_{i=0}^{n-1} (q^{2i+1} + 1)$ | generators through a point      |
| $H(2n+1, q^2)$, n even | $\prod_{i=0, i \neq n/2}^{n} (q^{2i+1} + 1)$ | No examples known               |
Our goal is to characterize the sets meeting the bounds.

- Is the point pencil the only possible construction for most of the polar spaces?
- For $Q^+(2n + 1, q)$, $n$ even, are the generators of one family the only possible construction?
- What can we say about $H(2n + 1, q^2)$, $n$ even?
A *d*-class *association scheme* on a finite set $\Omega$ is a pair $(\Omega, \mathcal{R})$ with $\mathcal{R}$ a set of symmetric relations $\{R_0, R_1, \ldots, R_d\}$ on $\Omega$ such that the following axioms hold:

(i) $R_0$ is the identity relation,
(ii) $\mathcal{R}$ is a partition of $\Omega^2$,
(iii) there are *intersection numbers* $p_{ij}^k$ such that for $(x, y) \in R_k$, the number of elements $z$ in $\Omega$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals $p_{ij}^k$.

All the relations $R_i$ are symmetric regular relations with valency $p_{ii}^0$, and hence define regular graphs on $\Omega$. 
Let $\Omega$ be the set of generators of the polar space $\mathcal{P}$. Two generators $\pi$ and $\pi'$ are adjacent iff they have empty intersection.

An EKR set of maximum size corresponds to a coclique of the graph of size $\frac{|\Omega|}{1-\frac{k}{r}}$.

If the dimension of a generator is $n$, then on $\Omega$ we can define a set of $n$ relations $\Gamma_i$, $i = 0, \cdots, n+1$ such that two generators are adjacent with respect to $\Gamma_i$ iff they intersect in a space of codimension $i$. These relations give rise to an association scheme.
Lemma

If $S$ is a subset of $\Omega$ such that its characteristic vector $\chi_S = h1 + u$, where $u$ is an eigenvector with eigenvalue $\lambda$ for the adjacency matrix $A_i$ of the relation $\Gamma_i$, then we have:

- every $p \in S$ has $\frac{|S|}{|\Omega|}(k - \lambda) + \lambda$ neighbors in $S$ w.r.t. $\Gamma_i$;
- every $p \notin S$ has $\frac{|S|}{|\Omega|}(k - \lambda)$ neighbors in $S$ w.r.t $\Gamma_i$;

where $k$ is the valency of the graph $\Gamma_i$.

The number of neighbors of $p$ depends only on the size of $S$. 
Most of the cases

For the following polar spaces:

- $Q(2n, q)$, $n$ even
- $Q^-(2n + 1, q)$
- $W(2n + 1, q)$, $n$ odd
- $H(2n, q^2)$ and $H(2n + 1, q^2)$, $n$ odd

If $u$ is an eigenvector for the relation $\Gamma_{n+1}$, then it is an eigenvector for $\Gamma_i, i = 0, \cdots, n$. 
Most of the cases

For every *EKR* set *S* of maximum size, we know how many elements of *S* intersect a fixed generator *π* in a space of codimension *i*, *i* = 1, ..., *n*: this number is a constant and it does not depend on the geometric structure of the set *S*.

**Known example of EKR in these polar spaces:**

The generators through a fixed point.

For every *π* ∈ *S*, the number of elements of *S* intersecting *π* in a space of codimension *i* is the same as the point pencil construction.

We focus on a fixed a generator of *S* and we get:

**Theorem**

For the polar spaces *Q*(2*n*, *q*), *n* even, *Q¬*(2*n* + 1, *q*), *W*(2*n* + 1, *q*), *n* odd, *H*(2*n*, *q*²) and *H*(2*n* + 1, *q*²), *n* odd, the largest *EKR* set of generators is the set of generators through a fixed point.
In $Q^+(2n + 1, q)$ there are two system of generators, $\Omega_1$ and $\Omega_2$ of the same size, such that two generators $\pi_1$ and $\pi_2$ are in the same system iff $\dim \pi_1 \cap \pi_2$ has the same parity as $n$.

**Even $n$**

The generators of $\Omega_i$ pairwise intersect in a non–empty space. The size of $\Omega_i$ meets the Stanton bound. It is the only possible EKR set meeting the bound.

**Odd $n$**

If $S$ is a maximum EKR set, then $S = S_1 \cup S_2$, where $S_i = S \cap \Omega_i$, $|S_1| = |S_2|$. If we find a EKR set $S_i$ in $\Omega_i$, $i = 1, 2$ and $|S_i| = \lfloor \frac{|S|}{2} \rfloor$, then $S_1 \cup S_2$ is a maximum EKR set in $\Omega$. 
**Q**(2n + 1, q), n odd

We can focus on only one system of generators \( \Omega_i \).

**Theorem**

If \( n > 3 \) is odd, then \( S_i \) is the set of elements of \( \Omega_i \) through a point. If \( n = 3 \), then \( S_i \) is either the set of elements of \( \Omega_i \) through a point or it is the set of elements of \( \Omega_i \) meeting a fixed element of \( \Omega_j \) in a plane.

**All generators: \( n > 3 \)**

We have two possibilities

- \( S \) is the set of all the generators through a point \( P \)
- \( S \) is the set of all the generators of one system through \( P_1 \) and the set of all the generators of the other system through \( P_2 \)
EKR Theorems for polar spaces
Overview of the results

\( Q^+(7, q) \)

We have four possibilities

- \( S \) is the set of all the solids through a point \( P \)
- \( S \) is the set of all the solids of one system through \( P_1 \) and the set of all the solids of the other system through \( P_2 \)
- \( S \) is the set of all solids of one system through \( P \) and all solids of the other system meeting \( \Sigma \) in a plane
- \( S \) is the set of all solids of one system meeting \( \Sigma_1 \) in a plane and all the generators meeting \( \Sigma_2 \) in a plane
Parabolic quadric $Q(2n, q), n$ odd

Embed $Q(2n, q), n$ odd, as a hyperplane section in a $Q^+(2n + 1, q)$: every generator of $Q(2n, q)$ is contained in a unique generator of a fixed system $\Omega_i$ of $Q^+(2n + 1, q)$.

An EKR set $S$ of maximum size of $Q(2n, q)$ gives rise to EKR set $S'$ of maximum size of $\Omega_i$.

**Theorem**

Let $Q(2n, q) = H \cap Q^+(2n + 1, q)$.
If $n > 3$, then $S'$ is a point pencil and we have two possibilities:

- $P \in H$, so $S$ is also a point pencil
- $P \notin H$, $S$ is the set of generators of one system of a $Q^+(2n - 1, q)$ embedded in $Q(2n, q)$.

If $n = 3$, then $S'$ can be a point pencil or the generators meeting a fixed one in a plane, so we have a third possibility:

- $S$ consists of the plane $\pi$ and all the planes meeting $\pi$ in a line
If $q$ is even, then:
\[ W(2n + 1, q) \cong Q(2n + 2, q) \]

There is a $Q^+(2n + 1, q)$ inducing the symplectic polarity

**Theorem**

An EKR set of maximum size $S$ is

- a point pencil
- the set of generators of one system of a $Q^+(2n + 1, q)$
- $n = 2$ and it consists of the plane $\pi$ and the planes meeting $\pi$ in a line
$W(2n + 1, q)$, $n$ even and $q$ odd

Let $v_{\pi,S}$ be the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$, then:

$$v = hv_1 + (1 - h)v_2$$

where $v_1$ arises from the point pencil construction and $v_2$ from the construction of the elements of one system of a hyperbolic quadric. Further investigation on the related association scheme and with more geometric arguments, we get:

**Theorem**

- $S$ is a point pencil or
- $n = 2$ and $S$ consists of the plane $\pi$ and the planes meeting $\pi$ in a line.
EKR set $|S| < \frac{|\Omega|}{1 - \frac{k}{\tau}} = \frac{|\Omega|}{q^{2n+1}+1}$ (more than point-pencil).

The algebraic combinatorial techniques cannot be used.

**Theorem for planes in $H(5, q^2)$**

- maximum size: $1 + q + q^3 + q^5 < \frac{|\Omega|}{q^3+1} = (q + 1)(q^5 + 1)$,
- only construction: a fixed plane and all the those meeting it in line.

If $S$ is a point pencil, then $|S| = (q + 1)(q^3 + 1) < 1 + q + q^3 + q^5$. 
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<td>$Q^+(7, q)$ a fixed system</td>
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<td>point pencil, a fixed plane and the planes meeting it in a line generators of one system in $Q^+(5, q)$ q even</td>
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<td>?</td>
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