Constructing *t*-designs with prescribed automorphism groups

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## Dear organizers, thank you very much for the invitation!



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# Dear organizers, thank you very much for the invitation!

This talk emerged unter the influence of my colleagues to whom I owe big thanks and want to mention in particular

- my close collaborators from the University of Zagreb, Croatia:
  - Vedran Krčadinac
  - Ivica Martinjak
  - Anamari Nakić
- as well as the colleagues from the University of Bayreuth, Germany:
  - Reinhard Laue
  - Axel Kohnert
  - Alfred Wassermann

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## Definition.

A t- $(v, k, \lambda)$  design  $\mathcal{D}$  is a pair  $(\mathcal{P}, \mathcal{B})$ , consisting of a v-element set of points  $\mathcal{P}$  and a collection  $\mathcal{B}$  of its k-element subsets called *blocks*, such that each t-element subset of  $\mathcal{P}$  is contained in exactly  $\lambda$  blocks.

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Every *t*-design is also an *s*-design, for all  $0 \le s \le t$ , with parameters

$$v_s = v, \quad k_s = k, \quad \lambda_s = \lambda \cdot \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$$

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$$v_s = v, \quad k_s = k, \quad \lambda_s = \lambda \cdot \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$$

In particular, each point is contained in

$$r = \lambda_1 = \lambda \cdot \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$$

blocks and since the empty set is contained in every block, the number of blocks equals to

$$b = \lambda_0 = \lambda \cdot \frac{\binom{v}{t}}{\binom{k}{t}}.$$

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## Examples.

2-(7,3,1) design

$$\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6\}$$
  
$$\mathcal{B} = \{\{1, 2, 3\}, \{0, 1, 4\}, \{0, 2, 5\}, \{0, 3, 6\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}\}$$

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3-(8,4,1) design

$$\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$
  

$$\mathcal{B} = \{\{1, 2, 3, 7\}, \{0, 1, 4, 7\}, \{0, 2, 5, 7\}, \{0, 3, 6, 7\}, \{1, 5, 6, 7\}, \{2, 4, 6, 7\}, \{3, 4, 5, 7\}, \{0, 4, 5, 6\}, \{2, 3, 5, 6\}, \{1, 3, 4, 6\}, \{1, 2, 4, 5\}, \{0, 2, 3, 4\}, \{0, 1, 3, 5\}, \{0, 1, 2, 6\}\}$$

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3-(8,4,1) design

$$\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$
  

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Besides as a list of blocks, a *t*-design can be represented by a 0-1 matrix  $M = [m_{ij}]$  called *incidence matrix*.

If we denote the points of  $\mathcal{D}$  by  $\mathcal{P} = \{p_1, p_2, \ldots, p_v\}$  and the blocks of  $\mathcal{D}$  by  $\mathcal{B} = \{B_1, B_2, \ldots, B_v\}$ , then the entries of the incidence matrix are defined by

$$m_{ij} = \left\{ \begin{array}{ll} 1 \,, \ \ {
m if} \ p_i \in B_j \ 0 \,, \ \ {
m otherwise} \end{array} 
ight.$$

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$$m_{ij} = \begin{cases} 1, & \text{if } p_i \in B \\ 0, & \text{otherwise} \end{cases}$$

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$$I = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

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0 1 1 1 0 0 0 1 0 0 0 1 1 1  $1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0$  $1 \ 1 \ 0$  $0 \ 0 \ 1 \ 0$ 

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For the sake of simplicity, rename the vertices of the cube:



0 0

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# $\mathbf{Goal}:$ construct $t\text{-}(v,k,\lambda)$ designs effectively

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construct  $t\text{-}(v,k,\lambda)$  designs effectively

#### Reasons:

- existence question (hard)
- enumeration problem (hard)
- get an example with given parameters quickly (not so hard)



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We shall consider two (similar) types of constraints:

- tactical decompositions
- automorphisms



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#### What is a tactical decomposition?

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#### What is a tactical decomposition?



$$M =$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

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#### What is a tactical decomposition?



$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Suppose that there is a partition

 $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \cdots \sqcup \mathcal{P}_m$  and  $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ ,

so that every submatrix  $M_{ij}$  of M, consisting of rows of  $\mathcal{P}_i$  and columns of  $\mathcal{B}_j$   $(i = 1, \ldots, m; j = 1, \ldots, n)$  has a constant number of 1's in each row and column. We shall call such a decomposition of the incidence matrix a *tactical decomposition*. ALCOMA

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Back FullScr Two tactical decompositions are trivial in case of a t- $(v, k, \lambda)$  design:

- the whole M itself (m = n = 1)
- m = v and n = b (each entry alone)

So, we look for non-trivial tactical decompositions!

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Two tactical decompositions are trivial in case of a t- $(v, k, \lambda)$  design:

- the whole M itself (m = n = 1)
- m = v and n = b (each entry alone)

So, we look for non-trivial tactical decompositions!

Denote by

 $\rho_{ij} = \text{the number of 1's in each row of } M_{ij}$  $\kappa_{ij} = \text{the number of 1's in each column of } M_{ij}.$ 

Further denote by

$$\begin{array}{ll} \langle p \rangle &= \{ B \in \mathcal{B} \mid p \in B \}, \text{ for any } p \in \mathcal{P} \text{ and} \\ \langle B \rangle &= \{ p \in \mathcal{P} \mid p \in B \}, \text{ for any } B \in \mathcal{B}, \end{array}$$

then we can formulate the coefficients  $ho_{ij}$  and  $\kappa_{ij}$  as

$$\rho_{ij} = |\langle p \rangle \cap \mathcal{B}_j|, \ p \in \mathcal{P}_i \\ \kappa_{ij} = |\langle B \rangle \cap \mathcal{P}_i|, \ B \in \mathcal{B}_j.$$

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#### In one of our previous examples



 $0 \ 1 | 1$ 0 | 1 $0 \ 1$ 0 1 1 0 0 1 1 0 0  $1 \ 1 \ 0$ 1 0 1 0 1 1 1 0 1 0 

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#### In one of our previous examples





these coefficients are:

$$[\rho_{ij}] = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 \end{bmatrix} \quad [\kappa_{ij}] = \begin{bmatrix} 4 & 0 & 2 & 2 & 2 & 2 \\ 0 & 4 & 2 & 2 & 2 & 2 \end{bmatrix}$$

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Find necessary conditions which a TD of a t- $(v, k, \lambda)$  has to fulfil!

Count the total number of 1's in  $M_{ij}$  in two ways.

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#### Find necessary conditions which a TD of a t- $(v, k, \lambda)$ has to fulfil!

Count the total number of 1's in  $M_{ij}$  in two ways.

You get:

$$|\mathcal{P}_i| \cdot 
ho_{ij} = |\mathcal{B}_j| \cdot \kappa_{ij}$$
 .

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(1)



#### Find necessary conditions which a TD of a t- $(v, k, \lambda)$ has to fulfil!

Count the total number of 1's in  $M_{ij}$  in two ways. You get:

$$|\mathcal{P}_i|\cdot
ho_{ij}=|\mathcal{B}_j|\cdot\kappa_{ij}$$
 .

Interprete it as a double counting of

$$\{(p,B) \mid p \in \mathcal{P}_i, B \in \mathcal{B}_j, p \in B\}.$$

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(1)



Take any point  $p \in \mathcal{P}_i$ . Look at the following set of triples:  $\{(p,q,B) \mid q \in \mathcal{P}_l, p \in B, q \in B\}$ 

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Take any point  $p \in \mathcal{P}_i$ . Look at the following set of triples:

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Take any point  $p \in \mathcal{P}_i$ . Look at the following set of triples:

 $\{(p,q,B) \mid q \in \mathcal{P}_l, \, p \in B, \, q \in B\}$ 

A double counting gives here the following equation:

$$\sum_{j=1}^n \rho_{ij}\kappa_{lj} = \sum_{q\in\mathcal{P}_l} |\langle \, p\rangle \, \cap \langle \, q\rangle \,| = \, \mathsf{I} \, \mathsf{know} \, \mathsf{that!}$$



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The generalized version of this formula is

$$\sum_{j=1}^{n} \rho_{ij} \kappa_{l_1 j} \kappa_{l_2 j} \cdots \kappa_{l_s j} =$$
$$\sum_{q_1 \in \mathcal{P}_{l_1}} \sum_{q_2 \in \mathcal{P}_{l_2}} \cdots \sum_{q_s \in \mathcal{P}_{l_s}} |\langle p \rangle \cap \langle q_1 \rangle \cap \langle q_2 \rangle \cap \cdots \cap \langle q_s \rangle |,$$

which one gets by taking a fixed point  $p \in \mathcal{P}_i$  and counting the set

$$\{(p, q_1, q_2, \dots, q_s, B) \mid q_1 \in \mathcal{P}_{l_1}, \dots, q_s \in \mathcal{P}_{l_s}, \\ p \in B, q_1 \in B, \dots, q_s \in B\}$$

in two different ways, for any appropriate integer s.



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(2)

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The generalized version of this formula is

$$\sum_{j=1}^{n} \rho_{ij} \kappa_{l_1 j} \kappa_{l_2 j} \cdots \kappa_{l_s j} =$$
$$\sum_{q_1 \in \mathcal{P}_{l_1}} \sum_{q_2 \in \mathcal{P}_{l_2}} \cdots \sum_{q_s \in \mathcal{P}_{l_s}} |\langle p \rangle \cap \langle q_1 \rangle \cap \langle q_2 \rangle \cap \cdots \cap \langle q_s \rangle |,$$

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in two different ways, for any appropriate integer s.

#### Good news: The right-hand side of (2) can be easily calculated in case of a *t*-design, for all $s \le t$ .

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(2)

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Are these equations all that I know about  $\rho_{ij}$  and  $\kappa_{ij}$ ?

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Are these equations all that I know about  $\rho_{ij}$  and  $\kappa_{ij}$ ?

Add the trivial conditions

$$\sum_{j=1}^n 
ho_{ij} = r, \ orall i, \ ext{and} \ \sum_{i=1}^m \kappa_{ij} = k, \ orall j.$$

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(3)

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Are these equations all that I know about  $\rho_{ij}$  and  $\kappa_{ij}$ ?

Add the trivial conditions

$$\sum_{j=1}^{n} \rho_{ij} = r, \quad \forall i, \text{ and } \sum_{i=1}^{m} \kappa_{ij} = k, \quad \forall j.$$
(3)

Together with (2), which can be rewritten via (1) only in terms of  $\rho_{ij}$ , that's it!

$$|\mathcal{P}_i| \cdot 
ho_{ij} = |\mathcal{B}_j| \cdot \kappa_{ij}$$

$$\sum_{j=1}^n 
ho_{ij}\kappa_{l_1j}\kappa_{l_2j}\cdots\kappa_{l_sj} =$$

$$\sum_{q_1 \in \mathcal{P}_{l_1}} \sum_{q_2 \in \mathcal{P}_{l_2}} \cdots \sum_{q_s \in \mathcal{P}_{l_s}} |\langle p \rangle \cap \langle q_1 \rangle \cap \langle q_2 \rangle \cap \cdots \cap \langle q_s \rangle$$

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## Another example of a tactical decomposition of a 3-(8, 4, 1) design:



Γ	0	1	1	1	0	0	0	1	0	0	0	1	1	1	1	0	1	1	1	0	0	0	1	0	0	0	1	1	17
I	1	1	0	0	1	0	0	0	0	1	1	0	1	1		1	1	0	0	1	0	0	0	0	1	1	0	1	1
	1	0	1	0	0	1	0	0	1	0	1	1	0	1		1	0	1	0	0	1	0	0	1	0	1	1	0	1
	1	0	0	1	0	0	1	0	1	1	0	1	1	0		1	0	0	1	0	0	1	0	1	1	0	1	1	0
l	0	1	0	0	0	1	1	1	0	1	1	1	0	0		0	1	0	0	0	1	1	1	0	1	1	1	0	0
	0	0	1	0	1	0	1	1	1	0	1	0	1	0		0	0	1	0	1	0	1	1	1	0	1	0	1	0
I	0	0	0	1	1	1	0	1	1	1	0	0	0	1		0	0	0	1	1	1	0	1	1	1	0	0	0	1
L	1	1	1	1	1	1	1	0	0	0	0	0	0	0		1	1	1	1	1	1	1	0	0	0	0	0	0	0

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What is an automorphism of a design?

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## What is an automorphism of a design? A well-known notion: a permutation of points which preserves the blocks.



 $\varphi = (0,1,2,3)(4,5,6,7)$ 

is an automorphism.  $\langle \varphi \rangle$  is an *automorphism group*, a subgroup of the *full automorphism group*  $Aut\mathcal{D}$ .

Point orbits  $\mathcal{P} = \{0, 1, 2, 3\} \sqcup \{4, 5, 6, 7\}$  partition  $\mathcal{P}$ Block orbits  $\mathcal{B} = \{\{0, 1, 2, 3\}\} \sqcup \{\{4, 5, 6, 7\}\} \sqcup$  other 4 sides  $\sqcup \{\{0, 2, 4, 6\}, \{1, 3, 5, 7\}\} \sqcup$  other 4 diag. par. edges  $\sqcup$  2 indep. sets partition  $\mathcal{B}$  ALCOMA

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**Proposition** Rows and columns of an incidence matrix M of a t- $(v, k, \lambda)$  design  $\mathcal{D}$  corresponding to the point and block orbits obtained under an action of an automorphism group  $G \leq Aut\mathcal{D}$  form a tactical decomposition.

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**Remark** Not every tactical decomposition of a t- $(v, k, \lambda)$  design comes from an action of an automorphism group of it! Our next constructions shall prove it.

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**Remark** Not every tactical decomposition of a t- $(v, k, \lambda)$  design comes from an action of an automorphism group of it! Our next constructions shall prove it.

There are (at least) two known different approaches how to construct designs with the additional constraint that an automorphism group acts on it:

- Finding all candidates for tactical decomposition matrices [ρ<sub>ij</sub>] and blowing them up to incidence matrices. (in addition: using this method, find designs with tactical decompositions which are not orbits)
- Finding the Kramer-Mesner matrix and solving the linear system of equations.

(in addition: improve the method by implementing the knowledge coming from the tactical decomposition matrices)

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Blowing up TDM's and getting designs without any automorphisms

## General construction procedure

- 1. Prescribe a group G.
- 2. Prescribe its action on points and blocks.
- 3. Classify all tactical decomposition matrix candidates, e.g. matrices  $[\rho_{ij}]$  fulfilling the equations (1) (3).
- 4. Try to construct incidence matrices of the designs, consistent with achieved TD matrices, by **forgetting** the group action at this stage, expanding each entry of the TD-matrix to its full size in an incidence matrix.



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Blowing up TDM's and getting designs without any automorphisms

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Since there is a computer program, written by V. Krčadinac, which solves the step 3 above quite general (for any *t*-design) we shall concentrate on step 4 - changing the coefficient  $\rho_{ij}$  by a 0-1 matrix with  $\rho_{ij}$  1's in each row and  $\kappa_{ij}$  1's in each column.



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Assume now that a cyclic group of order  $p, \ p$  prime, acts on a design. Then

 $|\mathcal{P}_i|, |\mathcal{B}_j| \in \{1, p\}$ 

The only interesting (non-unique) case for step 4 is

$$|\mathcal{P}_i| = |\mathcal{B}_j| = p.$$

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Note that Alltop's lemma gives immediately

$$\rho_{ij} = \kappa_{ij}$$

Hence, the problem in to replace the coefficient  $\rho_{ij}$  by a  $p \times p$  matrix  $M_{ij}$  possessing  $\rho_{ij}$  1's in each row and each column, taking care of the design properties.

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If you take  $M_{ij}$  to be cyclic, you preserve the group action, and have exactly

 $\begin{pmatrix} p \\ 0 \end{pmatrix}$ 

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If you want to forget the cyclic action, the number of possibilities becomes quite large very soon.

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If you want to forget the cyclic action, the number of possibilities becomes quite large very soon.

Throw a look at the table:

p	$ ho_{ij}$	$Cyc_{ij}$	$All_{ij}$
2	1	2	2
3	1	3	6
	2	3	6
5	1	5	120
	2	10	2040
	3	10	2040
	4	5	120
7	1	7	5040
	2	21	3110940
	3	35	68938800
	4	35	68938800
	5	21	3110940
	6	7	5040

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• In case p = 2: can't avoid the cyclic action!

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- In case p = 2: can't avoid the cyclic action!
- In case  $p \ge 5$ : the number of possibilities is quite large for an exhaustive search and comparison with the cyclic case.

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- In case p = 2: can't avoid the cyclic action!
- In case  $p \ge 5$ : the number of possibilities is quite large for an exhaustive search and comparison with the cyclic case.
- So, the convenient and interesting case is p = 3. Here, the number of possibilites in non-unique cases have doubled, and we can speak of the cyclic and anti-cyclic matrices of order 3 as replacements for  $\rho_{ij}$ , if  $\rho_{ij} \in \{1, 2\}$ , e.g. if  $\rho_{ij} = 1$ , then

- cyclic possibilities:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- anti-cyclic possibilities:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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The number of fixed points (and blocks)  $F \in \{0, 3, 6, 9\}$ .

F	TDM's
9	2
6	14
3	334
0	814



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9	2
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3	334
0	814

An exhaustive search of step 4 in the cyclic case gives

F	TDM's	niso
9	2	909
6	14	2368
3	334	79662
0	814	58720

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**Theorem** There are exactly 141061 symmetric designs with parameters (36, 15, 6) admitting an action of an automorphism of order 3.

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**Theorem** There are exactly 141061 symmetric designs with parameters (36, 15, 6) admitting an action of an automorphism of order 3.

Here is a complete list of the automorphism group orders and their frequencies among all constructed designs.

$ Aut(\mathcal{D}) $	3	6	9	12	18	21	24	27	30
niso	136733	3064	629	275	173	2	45	25	2
$ Aut(\mathcal{D}) $	36	42	48	54	72	81	96	108	144
niso	33	1	10	21	8	1	2	5	5
$ Aut(\mathcal{D}) $	162	216	240	243	324	360	384	432	486
niso	6	2	1	1	3	2	2	2	1
$ Aut(\mathcal{D}) $	648	1152	1944	3888	12096	51840			
niso	2	1	1	1	1	1			



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A (not quite exhaustive) search of step 4 in the **cyclic + anti-cyclic case** gives

F	niso Cyc	niso All
9	909	8176
6	2368	10885
3	79662	138149
0	58720	$\geq 509836$
all	141061	$\geq 665187$

This search was not exhaustive in case F = 0; the number of structures is quite large and we had to stop it.

Altogether, the following statement concludes this search:

**Proposition** There are at least 675363 symmetric designs with parameters (36, 15, 6) and 513692 of them don't admit any non-trivial automorphisms.

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The number of fixed points (and blocks)  $F \in \{5, 11\}$ .



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The number of fixed points (and blocks)  $F \in \{5, 11\}$ . Outcome of step 3:

 F
 TDM's

 11
 1

 5
 1834



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The number of fixed points (and blocks)  $F \in \{5, 11\}$ . Outcome of step 3:

 F
 TDM's

 11
 1

 5
 1834

An exhaustive search of step 4 in the cyclic case gives

F	TDM's	niso
11	1	3076
5	1834	342508



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The number of fixed points (and blocks)  $F \in \{5, 11\}$ . Outcome of step 3:

 F
 TDM's

 11
 1

 5
 1834

An exhaustive search of step 4 in the cyclic case gives

F	TDM's	niso
11	1	3076
5	1834	342508

**Theorem** There are exactly 345584 symmetric designs with parameters (41, 16, 6) admitting an action of an automorphism of order 3.



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The following table gives a concise overview of the described results for the cyclic case.

$ Aut(\mathcal{D}) $	F = 11	F = 5	all
3	2976	342241	345217
6	94	225	319
9	_	42	42
15	4	_	4
30	2	_	2
$\sum$	3076	342508	345584

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The following table gives a concise overview of the described results for the cyclic case.

$ Aut(\mathcal{D}) $	F = 11	F = 5	all
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$\sum$	3076	342508	345584

An exhaustive search in step 4 in the cyclic + anti-cyclic case gives

F	niso Cyc	niso All
11	3076	9808
5	342508	431276
all	345584	441048

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Altogether, the following statement concludes this search:

**Proposition** There are at least 441048 symmetric designs with parameters (41, 16, 6) and 95119 of them don't admit any non-trivial automorphisms.

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Altogether, the following statement concludes this search:

**Proposition** There are at least 441048 symmetric designs with parameters (41, 16, 6) and 95119 of them don't admit any non-trivial automorphisms.

We conclude this search with an informative table on the number of non-isomorphic (41, 16, 6) designs admitting only a tactical decomposition with partition sizes 1 and 3 and their automorphism group orders.

$ Aut(\mathcal{D}) $	F = 11	F = 5	all
1	6714	88423	95119
2		345	345
3	2994	342241	345217
6	94	225	319
9		42	42
15	4		4
30	2		2
$\sum$	9808	431276	441048

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G = group of permutations of  $\{1, 2, \ldots, v\}$ 



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G =group of permutations of  $\{1, 2, \dots, v\}$ 

 $\mathcal{T}_1$ ,  $\mathcal{T}_2$ ,  $\ldots$  ,  $\mathcal{T}_m$  orbits on t-subsets of  $\{1,\ldots,v\}$ 

 $\mathcal{K}_1$ ,  $\mathcal{K}_2$ ,  $\ldots$  ,  $\mathcal{K}_n$  orbits on k-subsets of  $\{1, \ldots, v\}$ 



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$$G =$$
group of permutations of  $\{1, 2, \dots, v\}$ 

 $\mathcal{T}_1$ ,  $\mathcal{T}_2$ ,  $\ldots$  ,  $\mathcal{T}_m$  orbits on t-subsets of  $\{1,\ldots,v\}$ 

 $\mathcal{K}_1$ ,  $\mathcal{K}_2$ , ...,  $\mathcal{K}_n$  orbits on k-subsets of  $\{1, \ldots, v\}$ 

 $a_{ij} = |\{ K \in \mathcal{K}_j \mid T \subseteq K \}|, \quad T \in \mathcal{T}_i \text{ (does not depend on choice!)}$  $A_{tk}^G = [a_{ij}] \quad \text{Kramer-Mesner matrix}$ 



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$$G =$$
group of permutations of  $\{1, 2, \dots, v\}$ 

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$$a_{ij} = |\{ K \in \mathcal{K}_j \mid T \subseteq K \}|, \quad T \in \mathcal{T}_i \text{ (does not depend on choice!)}$$
$$A_{tk}^G = [a_{ij}] \quad \text{Kramer-Mesner matrix}$$

**Theorem.** A simple t- $(v, k, \lambda)$  design with G as a group of automorphisms exists if and only if the system of linear equations  $A_{tk}^G \cdot x = \lambda j$  has a  $\{0, 1\}$ -solution.

The main **problem** when applying: the **size** of this linear system!



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**Main idea**: Incorporate the knowledge that gives you the tactical decomposition matrix when building the Kramer-Mesner matrix, to reduce the number of columns!

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**Main idea**: Incorporate the knowledge that gives you the tactical decomposition matrix when building the Kramer-Mesner matrix, to reduce the number of columns!

**Example** Have a look at the parameters 2-(7, 3, 1)

Define the point set to be  $P = \{0, 1, 2, 3, 4, 5, 6\}$ 

If we assume G to be trivial, its all orbits are of cardinality 1. Therefore, there are  $\binom{7}{2}$  orbits on the 2-element subsets of  $\mathcal{P}$  and  $\binom{7}{3}$  orbits on the 3-element subsets of  $\mathcal{P}$ .

We need to solve the system

$$A_{23} \cdot x = \lambda j \,.$$

**KM**-attempt: solve this  $21 \times 35$  system!

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**Main idea**: Incorporate the knowledge that gives you the tactical decomposition matrix when building the Kramer-Mesner matrix, to reduce the number of columns!

**Example** Have a look at the parameters 2-(7, 3, 1)

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We need to solve the system

$$A_{23} \cdot x = \lambda j \,.$$

**KM-attempt**: solve this  $21 \times 35$  system! Assume now

 $G=\langle\,(0)(1,2,3)(4,5,6)\rangle$ 

Now,  $A_{23}^G$  becomes a  $7 \times 13$  matrix.

**KM-attempt**: solve this  $7 \times 13$  system!

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Back FullScr What improvement gives our idea?

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### What improvement gives our idea?



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## What improvement gives our idea?

**Note**: only those k-orbits with columns in  $A_{1k}^G$  identical to a column of a TDM can be taken into further consideration - we have an additional necessary condition! Hence, **eliminate** the other k-orbits.

Plain KM  $A_{23}^G$ KM + TD  $A_{23}^G$  $7 \times 13$  $7 \times 7$ + 3 extra equations

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Back FullScr **Example** Parameters 3-(8, 4, 1)

 $\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ 

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$$G = \langle (0, 1, 2, 3)(4, 5, 6, 7) \}$$

**Example** Parameters 3-(8, 4, 1)

 $\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ 



$$G = \langle (0, 1, 2, 3)(4, 5, 6, 7) \}$$

**Plain KM:** a  $14 \times 20$  linear system.



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**Example** Parameters 3-(8, 4, 1) $\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ 



$$G = \langle (0, 1, 2, 3)(4, 5, 6, 7) \}$$

**Plain KM:** a  $14 \times 20$  linear system.

**KM** + **TD**: the action of G on blocks is not given, hence you have to check all possibilities. Only two lead to TD matrices:

- Block orbits [2, 4, 4, 4]The system of size  $14 \times 18 + 4$  additional equations Solution exists! (Which one is that?)
- Block orbits [1, 1, 2, 2, 4, 4]The system of size  $14 \times 12 + 4$  additional equations Solution exists!

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$$\mathcal{P} = \{0, 1, 2, \dots, 27\}$$
  

$$G = \langle \rho = (0, 1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12, 13)$$
  

$$(14, 15, 16, 17, 18, 19, 20)(21, 22, 23, 24, 25, 26, 27)$$
  

$$\sigma = (1, 6)(2, 5)(3, 4)(8, 13)(9, 12)(10, 11)$$
  

$$(15, 20)(16, 19)(17, 18)(22, 27)(23, 26)(24, 25)$$

So, I have chosen the dihedral group of order 14 to act on points in  $\left[7,7,7,7\right]\!.$ 

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$$\mathcal{P} = \{0, 1, 2, \dots, 27\}$$

$$G = \langle \rho = (0, 1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12, 13)$$

$$(14, 15, 16, 17, 18, 19, 20)(21, 22, 23, 24, 25, 26, 27),$$

$$\sigma = (1, 6)(2, 5)(3, 4)(8, 13)(9, 12)(10, 11)$$

$$(15, 20)(16, 19)(17, 18)(22, 27)(23, 26)(24, 25)$$

So, I have chosen the dihedral group of order 14 to act on points in  $\left[7,7,7,7\right]$ 

**Plain KM:** a  $36 \times 1533$  linear system.

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$$\mathcal{P} = \{0, 1, 2, \dots, 27\}$$

$$G = \langle \rho = (0, 1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12, 13)$$

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$$(15, 20)(16, 19)(17, 18)(22, 27)(23, 26)(24, 25)$$

So, I have chosen the dihedral group of order 14 to act on points in  $\left[7,7,7,7\right]$ 

**Plain KM:** a  $36 \times 1533$  linear system.

**KM** + **TD**: the action of G on blocks can be different, but only in orbits of sizes 7 and 14 - TD matrices exist only in 3 cases:

• Block orbits: [7,7,7,7,7,14,14] Number of TDM's: 1 The system contradictory ALCOMA

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- Block orbits: [7, 7, 7, 7, 7, 7, 7, 14] Number of TDM's: 6 System sizes: Ø, Ø, Ø, Ø, Ø, 44 × 97 Solution exists!
- Block orbits: [7,7,7,7,7,7,7,7,7] Number of TDM's: 7 System sizes: Ø, Ø, Ø, 45 × 36, 44 × 34, 45 × 36, 43 × 37 Solution exists for TDM's no. 5 and 7!



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- Block orbits: [7, 7, 7, 7, 7, 7, 7, 14] Number of TDM's: 6 System sizes: Ø, Ø, Ø, Ø, Ø, 44 × 97 Solution exists!
- Block orbits: [7,7,7,7,7,7,7,7,7] Number of TDM's: 7 System sizes: Ø, Ø, Ø, 45 × 36, 44 × 34, 45 × 36, 43 × 37 Solution exists for TDM's no. 5 and 7!

Comparison:

 Plain KM:
 1 linear system of size
  $36 \times 1533$ .

 KM + TD:
 5 linear systems of sizes:
  $44 \times 97$ ,

  $45 \times 36$ ,
  $44 \times 34$ ,

  $45 \times 36$ ,
  $43 \times 37$ .



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Choice of an automorphism group: the non-abelian group of order 39, acting on points in orbits of length [13, 13, 13, 13, 13].

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Choice of an automorphism group: the non-abelian group of order 39, acting on points in orbits of length [13, 13, 13, 13, 13].

**Plain KM:** I couldn't compute the KM-matrix in reasonable time! (Probably my fault, as it usually is the case...)

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**KM** + **TD**: the action of G on blocks can be different, but only in orbits of sizes 13 and 39. Here we shall take the usually hardest case - as many "long orbits" as possible.

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Choice of an automorphism group: the non-abelian group of order 39, acting on points in orbits of length [13, 13, 13, 13, 13].

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**KM** + **TD**: the action of G on blocks can be different, but only in orbits of sizes 13 and 39. Here we shall take the usually hardest case - as many "long orbits" as possible.

Take the block orbits to be [13, 39, 39, 39, 39, 39]. Number of TDM's: 1 System size:  $66 \times 351$ . (How doable!?!)

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Choice of an automorphism group: the non-abelian group of order 39, acting on points in orbits of length [13, 13, 13, 13, 13].

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**KM** + **TD**: the action of G on blocks can be different, but only in orbits of sizes 13 and 39. Here we shall take the usually hardest case - as many "long orbits" as possible.

Take the block orbits to be [13, 39, 39, 39, 39, 39]. Number of TDM's: 1 System size:  $66 \times 351$ . (How doable!?!) Solutions exist.

There are 10482 solution vectors as outcomes of the linear system solver. Only 263 designs are non-isomorphic.

For 262 the group G of order 39 is the full automorphism group and in one case  $|Aut\mathcal{D}| = 780$ .

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# Final comments

We use GAP as the background for all the computations. Hence, we have to add the limitations of GAP to our own limitations.



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# Final comments

We use GAP as the background for all the computations. Hence, we have to add the limitations of GAP to our own limitations.

Even with limitations, I hope to be able to run something for you, if you give me a limited problem, at any time of this conference, or after it.



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## Final comments

We use GAP as the background for all the computations. Hence, we have to add the limitations of GAP to our own limitations.

Even with limitations, I hope to be able to run something for you, if you give me a limited problem, at any time of this conference, or after it.

Thank you very much for you attention!



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