Extremal maximal isotropic codes of Type I-IV

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16.04.2010, Thurnau
Overview

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   - Extremality and a uniqueness result

3 Maximal self-orthogonal codes
   - Extremality for maximal self-orthogonal codes
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Let $F$ be a finite field, $N \in \mathbb{N}$. A code of length $N$ is a subspace $C \leq F^N$. 
Let $\mathbb{F}$ be a finite field, $N \in \mathbb{N}$. A \textit{code} of length $N$ is a subspace $C \leq \mathbb{F}^N$.

Let $\alpha$ be an automorphism of $\mathbb{F}$, of order 1 or 2. The \textit{dual} of $C$ is

$$C^\perp := \{ v \in \mathbb{F}^N \mid \sum_{i=1}^{N} v_i \cdot \alpha(c_i) = 0 \text{ for all } c \in C \},$$

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If $C \subseteq C^\perp$ then $C$ is called self-orthogonal.

If $C = C^\perp$ then $C$ is called self-dual.
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Due to the linearity of \( C \),

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d(C) = \min_{c \neq c' \in C} \left| \{ i \in \{1, \ldots, N\} \mid c_i \neq c'_i \} \right|.
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Using \(C\), one can

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Using \( C \), one can

- detect up to \( d(C) - 1 \) errors,
- correct up to \( \left\lfloor \frac{d(C)-1}{2} \right\rfloor \) errors.
The classical Types I-IV

Theorem (Gleason, Pierce 1967)

Let $C = C^\perp \leq \mathbb{F}_q^N$ and let $m \in \mathbb{N}$ such that $\text{wt}(c) \in m \mathbb{Z}$ for all $c \in C$. Then one of the following holds.
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(I) $q = 2$ and $m = 2$ (self-dual binary codes),
(II) $q = 2$ and $m = 4$ (doubly-even self-dual binary codes)
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(o) $q = 4$ and $m = 2$ (certain Euclidean self-dual codes)
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(o) \( q = 4 \) and \( m = 2 \) (certain Euclidean self-dual codes),
(d) \( m = 2 \) and \( C \cong \mathbb{F}^{N/2}_q \langle 1, a \rangle \), where either \( q \) is even and \( a = 1 \) or \( q \equiv 1 \pmod{4} \) and \( a^2 = -1 \) or \( \alpha \) has order 2 and \( a \cdot \alpha(a) = -1 \).
The first four Types in the previous theorem are named I-IV.
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Theorem

\textit{Let } \( T \in \{I, \ldots, IV\} \text{ and let } C \text{ be a self-dual Type } T \text{ code of length } N. \text{ Then } d(C) \leq \delta(T, N), \text{ where }

\[ \delta(T, N) := \begin{cases} 
     2 + 2\lfloor \frac{N}{8} \rfloor, & T = I \\
     4 + 4\lfloor \frac{N}{24} \rfloor, & T = II \\
     3 + 3\lfloor \frac{N}{12} \rfloor, & T = III \\
     2 + 2\lfloor \frac{N}{6} \rfloor, & T = IV.
\end{cases} \]
Introduction

Extremal self-dual codes of Type I-IV

Maximal self-orthogonal codes

Extremality and a uniqueness result

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Theorem

Let \( T \in \{I, \ldots, IV\} \) and let \( C \) be a self-dual Type \( T \) code of length \( N \). Then \( d(C) \leq \delta(T, N) \), where

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3 + 3\lfloor \frac{N}{12} \rfloor, & T = III \\
2 + 2\lfloor \frac{N}{6} \rfloor, & T = IV. 
\end{cases}
\]

If \( d(C) \) reaches the above bound then \( C \) is called extremal.
Extremality and a uniqueness result

We can read off $d(C)$ from the (Hamming) weight enumerator

$$\text{we}(C) := \sum_{c \in C} y^{\text{wt}(c)} x^{N - \text{wt}(c)} \in \mathbb{C}[x, y],$$

a homogeneous complex polynomial of degree $N$ which counts the codewords of each Hamming weight.
We can read off $d(C)$ from the *(Hamming) weight enumerator*

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a homogeneous complex polynomial of degree \( N \) which counts the codewords of each Hamming weight.

If \( C \) has minimum weight \( d \) then \( \text{we}(C) \) is of the form

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x^N + a_d y^d x^{N-d}
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If \( C \) has minimum weight \( d \) then \( \text{we}(C) \) is of the form

\[
x^N + a_d y^d x^{N-d} + \ldots + a_N y^N.
\]
Theorem

**If C is a self-dual Code of Type I, II, III or IV then we** \( (C) \in \mathbb{C}[f_T, g_T] \) **according to the table below.**

<table>
<thead>
<tr>
<th></th>
<th>( f_T )</th>
<th>( g_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( x^2 + y^2 )</td>
<td>( x^2 y^2 (x^2 - y^2)^2 ) Hamming code ( e_8 )</td>
</tr>
<tr>
<td></td>
<td>( i_2 )</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>( x^8 + 14x^4 y^4 + y^8 ) Hamming code ( e_8 )</td>
<td>( x^4 y^4 (x^4 - y^4)^4 ) binary Golay code ( g_{24} )</td>
</tr>
<tr>
<td>III</td>
<td>( x^4 + 8xy^3 ) tetracode ( t_4 )</td>
<td>( y^3(x^3 - y^3)^3 ) ternary Golay code ( g_{12} )</td>
</tr>
<tr>
<td>IV</td>
<td>( x^2 + 3y^2 ) ( i_2 \otimes \mathbb{F}_4 )</td>
<td>( y^2(x^2 - y^2)^2 ) hexacode ( h_6 )</td>
</tr>
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</table>
Extremality and a uniqueness result

Fix an integer $N$ and a Type $T \in \{I, \ldots, IV\}$ and let $\delta := \delta(T, N)$.
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$$x^N + a_\delta y^\delta x^{N-\delta} + \cdots + a_N y^N,$$

where $a_i \in \mathbb{Q}$ for $i = 1, \ldots, N$. 
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Using the Bürmann-Lagrange formula, one computes that $a_\delta \neq 0$.

**Corollary**

*The weight enumerator of an extremal self-dual code of Type I-IV is unique.*
The length of a self-dual Type $T$ code, $T \in \{I, \ldots, IV\}$, is always a multiple of

$$o_T := \deg(f_T) = \min(\{\deg(f_T), \deg(g_T)\})$$.
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Now assume that $N$ is no multiple of $o_T$. 
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Consider maximal self-orthogonal (m. s.-o.) codes, i.e. $C \subseteq C^\perp$ and if $C \subseteq D$ for a code $D \subseteq D^\perp$, then $C = D$.
Theorem

Let $C$ be a m. s.-o. Type II code of length $N \equiv 7 \pmod{8}$. Then $d(C^\perp) \leq 3 + 4\left\lfloor \frac{N+1}{24} \right\rfloor$. 
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Proof.

Assume that $d(C^\perp) \geq 4 + 4 \left\lfloor \frac{N+1}{24} \right\rfloor$. 

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Extremal self-dual codes of Type I-IV

Extremality for maximal self-orthogonal codes

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Let $C$ be a m. s.-o. Type II code of length $N \equiv 7 \pmod{8}$. Then $d(C^\perp) \leq 3 + 4 \left\lfloor \frac{N+1}{24} \right\rfloor$.

Proof.

Assume that $d(C^\perp) \geq 4 + 4 \left\lfloor \frac{N+1}{24} \right\rfloor$. From the theory of Witt groups, $C^\perp = \langle C, v \rangle$, where $\text{wt}(v) \equiv 3 \pmod{4}$. 

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Let $E = \begin{pmatrix} C & 0 \\ v & 1 \end{pmatrix} \leq \mathbb{F}_2^{N+1}$. Then $E = E^\perp$ is Type II, and
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Let \( E = \left( \begin{array}{cc} C & 0 \\ v & 1 \end{array} \right) \leq \mathbb{F}_2^{N+1} \). Then \( E = E^\perp \) is Type II, and

\[ d(E) \geq 4 + 4\left\lfloor \frac{N+1}{24} \right\rfloor, \]

hence \( E \) is extremal (i.e. equality holds). Thus the words in \( E \) of weight \( d(E) \) hold a design.
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Let $C$ be a m. s.-o. Type II code of length $N \equiv 7 \pmod{8}$. Then $d(C^\perp) \leq 3 + 4\lfloor \frac{N+1}{24} \rfloor$.

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- $d(E) \geq 4 + 4\lfloor \frac{N+1}{24} \rfloor$, hence $E$ is extremal (i.e. equality holds). Thus the words in $E$ of weight $d(E)$ hold a design.
- $\{ e \in E \mid \text{wt}(e) = d(E) \} = \{(c 0) \mid c \in C^\perp, \text{wt}(c) = d(E)\}$. 

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Extremal maximal isotropic codes of Type I-IV
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- $\{ e \in E \mid \text{wt}(e) = d(E) \} = \{(c \ 0) \mid c \in C^\perp, \text{wt}(c) = d(E)\}$.

This is a contradiction, hence $d(C^\perp) \leq 3 + 4\left\lfloor \frac{N+1}{24} \right\rfloor$. \qed
Theorem

Let $T \in \{I, \ldots, IV\}$ and let $C$ be a maximal self-orthogonal Type $T$ code of length $N$. Then $d(C^\perp) \leq \delta(T, N)$, where $\delta(T, N)$ is given in the table below.
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Let $T \in \{I, \ldots, IV\}$ and let $C$ be a maximal self-orthogonal Type $T$ code of length $N$. Then $d(C^\perp) \leq \delta(T, N)$, where $\delta(T, N)$ is given in the table below.

Definition

A m. s.-o. code whose minimum distance reaches the above bound is called dual extremal.
Extremal maximal isotropic codes of Type I-IV

### Extremal self-dual codes of Type I-IV

<table>
<thead>
<tr>
<th>$T$</th>
<th>$N$</th>
<th>$\delta(T, N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$N \neq 24$ 23</td>
<td>$\delta(I, N + 1)$</td>
</tr>
<tr>
<td></td>
<td>23 (24)</td>
<td>$3 + 4 \lfloor \frac{N}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>1, 9 or 17 (24)</td>
<td>$1 + \lfloor \frac{N}{24} \rfloor + 3 \lfloor \frac{N+7}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>2 (24)</td>
<td>$\lfloor \frac{N+8}{6} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>3, 11 or 19 (24)</td>
<td>$1 + 2 \lfloor \frac{N}{24} \rfloor + \lfloor \frac{N+5}{24} \rfloor + \lfloor \frac{N+13}{24} \rfloor$</td>
</tr>
<tr>
<td>II</td>
<td>4 (24)</td>
<td>$\frac{N+8}{6}$</td>
</tr>
<tr>
<td></td>
<td>5 (24)</td>
<td>$1 + 4 \lfloor \frac{N}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>6 (24)</td>
<td>$2 + 4 \lfloor \frac{N}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>7, 13, 14 or 15 (24)</td>
<td>$3 + 4 \lfloor \frac{N}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>10 or 18 (24)</td>
<td>$1 + \lfloor \frac{N}{8} \rfloor + \lfloor \frac{N+8}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>12 (24)</td>
<td>$\frac{N}{6}$</td>
</tr>
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### Maximal self-orthogonal codes

<table>
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<th>$N$</th>
<th>$\delta(T, N)$</th>
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<tbody>
<tr>
<td></td>
<td>20 (24)</td>
<td>$\frac{N+4}{6}$</td>
</tr>
<tr>
<td></td>
<td>21 (24)</td>
<td>$5 + 4 \lfloor \frac{N}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>22 (24)</td>
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<td></td>
<td>23 (24)</td>
<td>$7 + 4 \lfloor \frac{N}{24} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>1, 5 or 9 (12)</td>
<td>$3 + 3 \lfloor \frac{N}{12} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>2 (12)</td>
<td>$1 + 3 \lfloor \frac{N}{12} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>3, 6 or 7 (12)</td>
<td>$2 + 3 \lfloor \frac{N}{12} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>10 (12)</td>
<td>$4 + 3 \lfloor \frac{N}{12} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>11 (12)</td>
<td>$5 + 3 \lfloor \frac{N}{12} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>1 or 3 (6)</td>
<td>$1 + 2 \lfloor \frac{N}{6} \rfloor$</td>
</tr>
<tr>
<td></td>
<td>5 (6)</td>
<td>$3 + 2 \lfloor \frac{N}{6} \rfloor$</td>
</tr>
</tbody>
</table>
A uniqueness result

Theorem

*The Hamming weight enumerator of a dual extremal m. s.-o. code of Type II, III or IV is uniquely determined.*
A uniqueness result

**Theorem**

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What is the algebraic structure of the vector space generated by weight enumerators of m. s.-o. codes of Type I-IV?
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What is the algebraic structure of the vector space generated by weight enumerators of m. s.-o. codes of Type I-IV?

Definition

For $T \in \{I, \ldots, IV\}$ and $k \in \{1, \ldots, o_T - 1\}$, let

$$I^T_k := \langle \text{we}(C) \mid C \text{ m. s.-o. Type } T \text{ code of length } \equiv k(\text{mod } o_T) \rangle \subset \mathbb{C}.$$
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Let $C$ be a m. s.-o. Type $T$ code of length $\equiv k \pmod{o_T}$, and let $D$ be a self-dual Type $T$ code.
Theorem

The Hamming weight enumerator of a dual extremal m. s.-o. code of Type II, III or IV is uniquely determined.

What is the algebraic structure of the vector space generated by weight enumerators of m. s.-o. codes of Type I-IV?

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For $T \in \{I, \ldots, IV\}$ and $k \in \{1, \ldots, o_T - 1\}$, let

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Let $C$ be a m. s.-o. Type $T$ code of length $\equiv k \pmod{o_T}$, and let $D$ be a self-dual Type $T$ code.

Then $C \perp D$ is a m. s.-o. Type $T$ code of length $\equiv k \pmod{o_T}$. 
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**Extremal maximal isotropic codes of Type I-IV**
A uniqueness result

Remark

The space $I_k^T$ is a module for $\mathbb{C}[f_T, g_T]$. 

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A uniqueness result

**Remark**

_The space \( l_k^T \) is a module for \( \mathbb{C}[f_T, g_T] \)._ 

**Theorem**

_The \( \mathbb{C}[f_T, g_T] \)-module \( l_k^T \) is free and finitely generated._
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The space $I_k^T$ is a module for $\mathbb{C}[f_T, g_T]$.

Theorem

The $\mathbb{C}[f_T, g_T]$-module $I_k^T$ is free and finitely generated.

Bases for the $\mathbb{C}[f_T, g_T]$-module $I_k^T$ are given in the book "Self-dual codes and invariant theory" by Nebe, Rains and Sloane.
Remark
The space $I^T_k$ is a module for $\mathbb{C}[f_T, g_T]$.

Theorem
The $\mathbb{C}[f_T, g_T]$-module $I^T_k$ is free and finitely generated.

Bases for the $\mathbb{C}[f_T, g_T]$-module $I^T_k$ are given in the book "Self-dual codes and invariant theory" by Nebe, Rains and Sloane. There exists a triangular basis $p_0, \ldots, p_r$ of

$$(I^T_k)_N := \{p \in I^T_k \mid p \text{ homogeneous of degree } N\},$$

for every integer $N \equiv k \pmod{o_T}$. 

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A uniqueness result

\[ p_i(1, y) = c_i^{(0)} y^0 + \ldots + c_i^{(N)} y^N \]
A uniqueness result

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\[
\begin{array}{cccccccc}
y^0 & y^1 & \ldots & y^k & y^{k+1} & y^{k+2} & \ldots \\
p_0 & c_0^{(0)} & c_1^{(0)} & \ldots & c_k^{(0)} & 0 & c_{k+2}^{(0)} & \ldots \\
p_1 & 0 & c_1^{(1)} & \ldots & c_k^{(1)} & 0 & c_{k+2}^{(1)} & \ldots \\
p_k & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
p_{k+1} & 0 & \ldots & \ldots & 0 & c_{k+2}^{(k+2)} & \ldots \\
p_k & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \vdots \\
\end{array}
\]
If \( T \in \{\text{II, III, IV}\} \) and \( N \equiv -1 \pmod{o_T} \) then puncturing an extremal self-dual code of length \( N + 1 \) yields the dual of a dual extremal m. s.-o. code of length \( N \).
If $T \in \{\text{II}, \text{III}, \text{IV}\}$ and $N \equiv -1 \pmod{o_T}$ then puncturing an extremal self-dual code of length $N + 1$ yields the dual of a dual extremal m. s.-o. code of length $N$.

- The dual of the binary $[7, 4, 3]$ Hamming code is the unique dual extremal Type II code of length 7.
Examples

If $T \in \{\text{II, III, IV}\}$ and $N \equiv -1 \pmod{o_T}$ then puncturing an extremal self-dual code of length $N + 1$ yields the dual of a dual extremal m. s.-o. code of length $N$.

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Extremal maximal isotropic codes of Type I-IV
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This is false for \( T = I \) and \( N = 17 \), e.g. \( \delta(I, 18) = 4 = \delta(1, 17) \). Let \( C, D \) be the two extremal self-dual \([18, 9, 4]\) codes.
If $T \in \{\text{II, III, IV}\}$ and $N \equiv -1 \pmod{o_T}$ then puncturing an extremal self-dual code of length $N + 1$ yields the dual of a dual extremal m. s.-o. code of length $N$.

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Let $C, D$ be the two extremal self-dual $[18, 9, 4]$ codes.

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If $T \in \{\text{II, III, IV}\}$ and $N \equiv -1 \pmod{o_T}$ then puncturing an extremal self-dual code of length $N + 1$ yields the dual of a dual extremal m. s.-o. code of length $N$.

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Let $C, D$ be the two extremal self-dual $[18, 9, 4]$ codes.

- Puncturing $C$ at a particular position yields the dual of a dual extremal $[17, 8]$ code.
- Puncturing $D$ at any position yields codes of minimum weight 3.