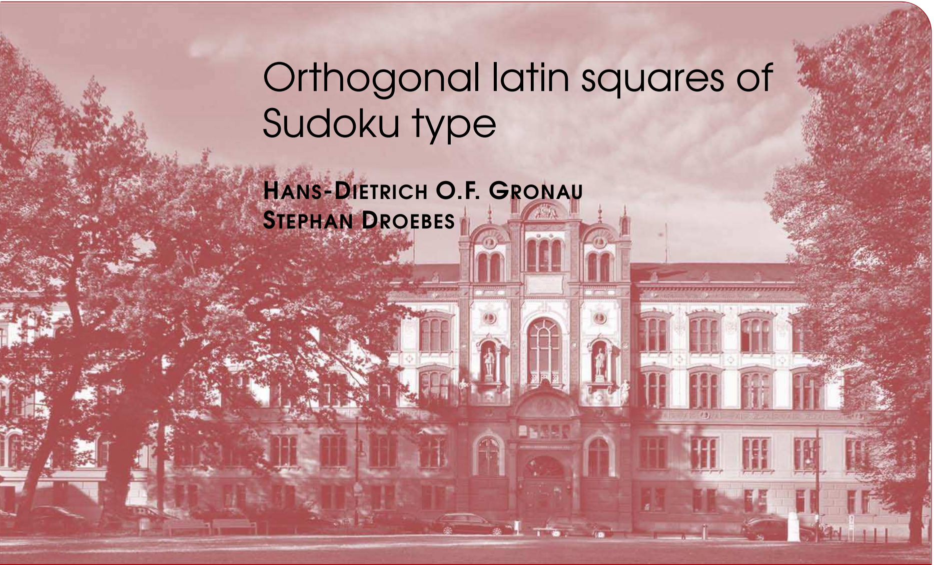
Universität Rostock

Traditio et Innovatio

Orthogonal latin squares of Sudoku type

Hans-Dietrich O.F. Gronau
Stephan Droebes



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First example

	3							
			1	9	5			
	9	8					6	
8				6				
4					3			1
				2				
	6					2	8	
			4	1	9			5
							7	

Theorem

There are exactly 5,524,751,496,156,892,842,531,225,600 different latin squares of order 9.

Theorem (Felgenhauer & Jarvis 2005)

There are exactly 6,670,903,752,021,072,036,060 different latin squares of order 9 of Sudoku type.

Theorem (Russell & Jarvis)

There are exactly 5,472,730,538 non-isomorphic latin squares of order 9 of Sudoku type.

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Theorem (Russell & Jarvis)

There are exactly 5,472,730,538 non-isomorphic latin squares of order 9 of Sudoku type.

Latin square of order 3×4 of Sudoku type

1	2	3	4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	1	2	3	4
9	10	11	12	1	2	3	4	5	6	7	8
2	3	4	1	6	7	8	5	10	11	12	9
6	7	8	5	10	11	12	9	2	3	4	1
10	11	12	9	2	3	4	1	6	7	8	5
3	4	1	2	7	8	5	6	11	12	9	10
7	8	5	6	11	12	9	10	3	4	1	2
11	12	9	10	3	4	1	2	7	8	5	6
4	1	2	3	8	5	6	7	12	9	10	11
8	5	6	7	12	9	10	11	4	1	2	3
12	9	10	11	4	1	2	3	8	5	6	7

Gerechte latin squares

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

Gerechte latin squares

?	2	3	4	5	6	...	n
1						...	
						...	
...	...	...	...	...	...	...	...

Gerechte latin squares

L1=

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4

Some references

1. R.A. Bailey, P.J. Cameron ans R. Connelly, Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes, *Amer. Math. Monthly* **115** (2008), 383-404.
2. E.R. Vaughan, The complexity of contructing gerechte designs, *The electronic Journal of Combinatorics* **16** (2009), R15;
   also 40th and 41th CGTC, Boca Raton, FL, U.S.

Minimal Sudoku

							1	
4								
	2							
				5		4		7
		8				3		
		1		9				
3			4			2		
	5		1					
			8		6			

First example by Paul Vaderlind

59			18			84		
	2				94		43	
1		74		63				98
6	9		5	7	2	6		1
6		2	2		5	9		1
4		1	6	3	3		9	2
93				15		31		6
	85		91				9	
		47			69			25

First example by Paul Vaderlind - completed

59	23	96	18	42	75	84	61	37
88	62	35	57	21	94	16	43	79
17	41	74	89	63	36	55	22	98
26	99	53	45	78	12	67	34	81
65	38	82	24	97	51	49	76	13
44	77	11	66	39	83	28	95	52
93	56	29	72	15	48	31	87	64
32	85	68	91	54	27	73	19	46
71	14	47	33	86	69	92	58	25

First example by Paul Vaderlind - completed

5	2	9	1	4	7	8	6	3
8	6	3	5	2	9	1	4	7
1	4	7	8	6	3	5	2	9
2	9	5	4	7	1	6	3	8
6	3	8	2	9	5	4	7	1
4	7	1	6	3	8	2	9	5
9	5	2	7	1	4	3	8	6
3	8	6	9	5	2	7	1	4
7	1	4	3	8	6	9	5	2

9	3	6	8	2	5	4	1	7
8	2	5	7	1	4	6	3	9
7	1	4	9	3	6	5	2	8
6	9	3	5	8	2	7	4	1
5	8	2	4	7	1	9	6	3
4	7	1	6	9	3	8	5	2
3	6	9	2	5	8	1	7	4
2	5	8	1	4	7	3	9	6
1	4	7	3	6	9	2	8	5

Definitions

1. $N(n) = \max\{k : \exists k \text{ MOLS of order } n\}$
2. $N_S(n) = \max\{k : \exists k \text{ MOLS of order } n \text{ of Sudoku type}\}$
3. $N_S(a \times b) = \max\{k :$
   $\exists k \text{ MOLS of order } ab \text{ of Sudoku type in } a \times b - rectangles\}$

Definitions

1. $N(n) = \max\{k : \exists k$ MOLS of order $n\}$
2. $N_S(n) = \max\{k : \exists k$ MOLS of order n of Sudoku type$\}$
3. $N_S(a \times b) = \max\{k :$
   $\exists k$ MOLS of order ab of Sudoku type in $a \times b - rectangles\}$

Definitions

1. $N(n) = \max\{k : \exists k \text{ MOLS of order } n\}$
2. $N_S(n) = \max\{k : \exists k \text{ MOLS of order } n \text{ of Sudoku type}\}$
3. $N_S(a \times b) = \max\{k :$
   $\exists k \text{ MOLS of order } ab \text{ of Sudoku type in } a \times b - rectangles\}$

Bounds

Theorem

$N(n) \leq n - 1.$

Theorem

$N_s(n^2) \leq n^2 - n.$

Theorem

$N_s(a \times b) \leq ab - a.$
$N_s(a \times b) \leq ab - b.$

Bounds

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$N(n) \leq n - 1.$

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Bounds

Theorem

$N(n) \leq n - 1.$

Theorem

$N_s(n^2) \leq n^2 - n.$

Theorem

$N_s(a \times b) \leq ab - a.$
$N_s(a \times b) \leq ab - b.$

Bounds - Proof

1	2	...	n	...	n^2-n+1	n^2-n+2	...	n^2
?				...			...	
				...			...	
...	...	...	...	...	...	...	...	...

$N_S(n)$ for small n

n	2	3	4	5	6	7	8	9	10
$N(n)$	1	2	3	4	1	6	7	8	$? \geq 2, \leq 8$
$N_S(n)$	-	-	2	-	1	-	4	6	$? \geq 1, \leq 5$

$N_S(n) \geq 1$

1	2	3	4	5	6	7	8	9	0
6	7	8	9	0	1	2	3	4	5
5	1	2	3	4	0	6	7	8	9
0	6	7	8	9	5	1	2	3	4
4	5	1	2	3	9	0	6	7	8
9	0	6	7	8	4	5	1	2	3
3	4	5	1	2	8	9	0	6	7
8	9	0	6	7	3	4	5	1	2
2	3	4	5	1	7	8	9	0	6
7	8	9	0	6	2	3	4	5	1

$N_S(4) = 2$

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

1	2	3	4
4	3	2	1
3	4	1	2
2	1	4	3

$N_S(2 \times 4) = 4$

L1=

1	2	3	4	5	6	7	8
5	6	7	8	1	2	3	4
4	3	2	1	8	7	6	5
8	7	6	5	4	3	2	1
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
6	5	8	7	2	1	4	3
2	1	4	3	6	5	8	7

L2=

1	2	3	4	5	6	7	8
6	5	8	7	2	1	4	3
2	1	4	3	6	5	8	7
5	6	7	8	1	2	3	4
3	4	1	2	7	8	5	6
8	7	6	5	4	3	2	1
4	3	2	1	8	7	6	5
7	8	5	6	3	4	1	2

L3=

1	2	3	4	5	6	7	8
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
6	5	8	7	2	1	4	3
4	3	2	1	8	7	6	5
3	4	1	2	7	8	5	6
5	6	7	8	1	2	3	4

L4=

1	2	3	4	5	6	7	8
8	7	6	5	4	3	2	1
6	5	8	7	2	1	4	3
3	4	1	2	7	8	5	6
2	1	4	3	6	5	8	7
7	8	5	6	3	4	1	2
5	6	7	8	1	2	3	4
4	3	2	1	8	7	6	5

$N_S(9) = 6$

L1=

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4

L2=

1	2	3	4	5	6	7	8	9
5	6	4	8	9	7	2	3	1
9	7	8	3	1	2	6	4	5
6	4	5	9	7	8	3	1	2
7	8	9	1	2	3	4	5	6
2	3	1	5	6	4	8	9	7
8	9	7	2	3	1	5	6	4
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	1	2	3

L3=

1	2	3	4	5	6	7	8	9
6	4	5	9	7	8	3	1	2
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5
2	3	1	5	6	4	8	9	7
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
7	8	9	1	2	3	4	5	6
3	1	2	6	4	5	9	7	8

L4=

1	2	3	4	5	6	7	8	9
7	8	9	1	2	3	4	5	6
4	5	6	7	8	9	1	2	3
2	3	1	5	6	4	8	9	7
8	9	7	2	3	1	5	6	4
5	6	4	8	9	7	2	3	1
3	1	2	6	4	5	9	7	8
9	7	8	3	1	2	6	4	5
6	4	5	9	7	8	3	1	2

L5=

1	2	3	4	5	6	7	8	9
8	9	7	2	3	1	5	6	4
6	4	5	9	7	8	3	1	2
5	6	4	8	9	7	2	3	1
3	1	2	6	4	5	9	7	8
7	8	9	1	2	3	4	5	6
9	7	8	3	1	2	6	4	5
4	5	6	7	8	9	1	2	3
2	3	1	5	6	4	8	9	7

L6=

1	2	3	4	5	6	7	8	9
9	7	8	3	1	2	6	4	5
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4
4	5	6	7	8	9	1	2	3
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
2	3	1	5	6	4	8	9	7
7	8	9	1	2	3	4	5	6

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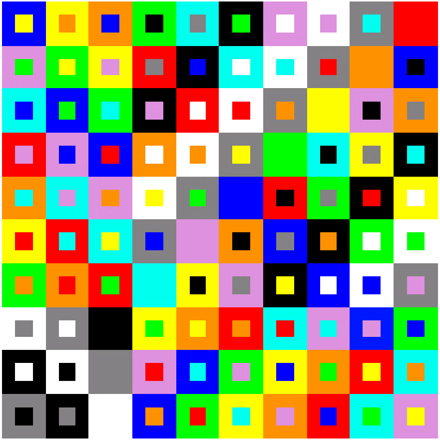
$N_S(9) = 6$ in complete sets of 8 MOLS of order 9

Set 1 (Plane Φ)							
123456789	123456789	123456789	123456789	123456789	123456789	123456789	123456789
231564897	456789123	645978312	897231564	312645978	564897231	789123456	978312645
312645978	789123456	897231564	645978312	231564897	978312645	456789123	564897231
456789123	312645978	978312645	564897231	789123456	645978312	231564897	897231564
564897231	645978312	231564897	312645978	978312645	789123456	897231564	456789123
645978312	978312645	456789123	789123456	897231564	231564897	564897231	312645978
789123456	231564897	564897231	978312645	456789123	897231564	312645978	645978312
897231564	564897231	789123456	456789123	645978312	312645978	978312645	231564897
978312645	897231564	312645978	231564897	564897231	456789123	645978312	789123456
a,0	a,0	a,0	a,0	a,0	a,0	a,0	a,0

Set 2 (Plane Ω, $t = \ell_\infty$)							
123456789	123456789	123456789	123456789	123456789	123456789	123456789	123456789
231564897	456789123	645978312	564897231	312645978	789123456	897231564	978312645
312645978	789123456	897231564	978312645	231564897	456789123	645978312	564897231
456789123	312645978	564897231	897231564	789123456	231564897	978312645	645978312
564897231	645978312	789123456	312645978	978312645	897231564	456789123	231564897
645978312	978312645	312645978	456789123	897231564	564897231	231564897	789123456
789123456	231564897	978312645	645978312	456789123	312645978	564897231	897231564
897231564	564897231	231564897	789123456	645978312	978312645	312645978	456789123
978312645	897231564	456789123	231564897	564897231	645978312	789123456	312645978
a,0	a,0	a,0	a,0	a,0	a,0	a,0	a,0

$N(10) \geq 2$



MacNeish

Theorem

For all $m, n \geq 2$*:* $N_S((mn)^2) \geq \min\{N_S(m^2), N_S(n^2)\}$.

Example

$N_S(36) \geq \min\{N_S(9), N_S(4)\} = \min\{6, 2\} = 2.$

MacNeish

Theorem

For all $m, n \geq 2$*:* $N_S((mn)^2) \geq \min\{N_S(m^2), N_S(n^2)\}$.

Example

$N_S(36) \geq \min\{N_S(9), N_S(4)\} = \min\{6, 2\} = 2.$

n is the square of a prime power

Theorem

Let n be a prime power. Then $N_S(n^2) = n^2 - n$.

$N_S(16) = 12$

Spalte / Zeile	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	5	6	7	8	9	10	11	12	13	14	15	16
3	9	11	12	10	13	15	16	14	5	7	8	6
4	13	16	14	15	5	8	6	7	9	12	10	11
5	7	3	15	11	12	16	4	8	14	10	6	2
6	3	8	9	14	4	7	10	13	2	5	12	15
7	15	9	6	4	8	2	13	11	10	16	3	5
8	11	14	4	5	16	9	7	2	6	3	13	12
9	12	4	8	16	14	6	2	10	7	15	11	3
10	16	7	2	9	6	13	12	3	11	4	5	14
11	4	10	13	7	2	12	15	5	3	9	14	8
12	8	13	11	2	10	3	5	16	15	6	4	9
13	14	2	10	6	7	11	3	15	12	8	16	4
14	10	5	16	3	15	4	9	6	8	11	2	13
15	6	12	3	13	11	5	14	4	16	2	9	7
16	2	15	5	12	3	14	8	9	4	13	7	10

Proof:

Let
$\mathbb{F}_q$,
$\mathbb{F}_{q^2} = \{x + \alpha y : x, y \in \mathbb{F}_q\}$ and
$\widetilde{\mathbb{F}_{q^2}} = \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then the latin squares defined by

$$L_x(i,j) = ix + j; x \in \widetilde{\mathbb{F}_{q^2}}; i,j \in \mathbb{F}_{q^2}.$$

are of Sudoku type.

Corollary

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *and* $q = \min\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}\}$. *Then* $N_S(n^2) \geq q^2 - q$.

Corollary

Let $n \geq 2$. *Then* $N_S(n^2) \geq 2$.

Corollary

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *and* $q = \min\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}\}$. *Then* $N_S(n^2) \geq q^2 - q$.

Corollary

Let $n \geq 2$. *Then* $N_S(n^2) \geq 2$.

Theorem

Let n be a prime power satisfying $q = p^r = ab, a = p^s, b = p^t$ and $t \geq s$. Then $N_S(a \times b) = ab - b$.

$N_S(2 \times 4) = 4$

L1=

1	2	3	4	5	6	7	8
5	6	7	8	1	2	3	4
4	3	2	1	8	7	6	5
8	7	6	5	4	3	2	1
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
6	5	8	7	2	1	4	3
2	1	4	3	6	5	8	7

L2=

1	2	3	4	5	6	7	8
6	5	8	7	2	1	4	3
2	1	4	3	6	5	8	7
5	6	7	8	1	2	3	4
3	4	1	2	7	8	5	6
8	7	6	5	4	3	2	1
4	3	2	1	8	7	6	5
7	8	5	6	3	4	1	2

L3=

1	2	3	4	5	6	7	8
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
6	5	8	7	2	1	4	3
4	3	2	1	8	7	6	5
3	4	1	2	7	8	5	6
5	6	7	8	1	2	3	4

L4=

1	2	3	4	5	6	7	8
8	7	6	5	4	3	2	1
6	5	8	7	2	1	4	3
3	4	1	2	7	8	5	6
2	1	4	3	6	5	8	7
7	8	5	6	3	4	1	2
5	6	7	8	1	2	3	4
4	3	2	1	8	7	6	5

Theorem

Let $m, n \geq 2$. *Then* $N_S(m \times n) \geq \min\{N(m), N(n)\}$.

Example

$$N_S(3 \times 4) \geq min\{N(3), N(4)\} = min\{2, 3\} = 2.$$

Let $A^{(i)}$ and $B^{(i)}$ for $i \in \{1, 2\}$ be two MOLS of order 3 and 4.

$A^{(1)} =$

0	1	2
1	2	0
2	0	1

, $A^{(2)} =$

0	1	2
2	0	1
1	2	0

$B^{(1)} =$

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

, $B^{(2)} =$

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

Example

$$N_S(3 \times 4) \geq min\{N(3), N(4)\} = min\{2,3\} = 2.$$

Let $A^{(i)}$ and $B^{(i)}$ for $i \in \{1,2\}$ be two MOLS of order 3 and 4.

$A^{(1)} =$

0	1	2
1	2	0
2	0	1

, $A^{(2)} =$

0	1	2
2	0	1
1	2	0

$B^{(1)} =$

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

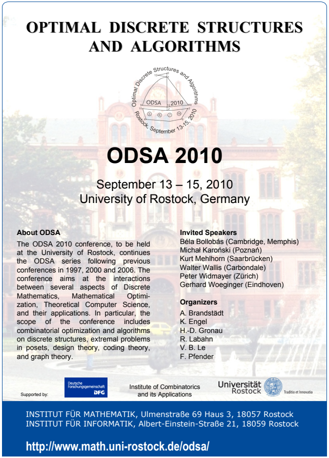
, $B^{(2)} =$

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

Then $C^{(1)} = A^{(1)} \otimes B^{(1)}$ and $C^{(2)} = A^{(2)} \otimes B^{(2)}$.

$$
C^{(1)} = \begin{array}{|cccc|cccc|cccc|}
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 0 & 1 & 2 & 3 \\
8 & 9 & 10 & 11 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 \\
7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 & 3 & 2 & 1 & 0 \\
11 & 10 & 9 & 8 & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
\hline
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 \\
5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 & 1 & 0 & 3 & 2 \\
9 & 8 & 11 & 10 & 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
\hline
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 \\
6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 & 2 & 3 & 0 & 1 \\
10 & 11 & 8 & 9 & 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
\hline
\end{array}
$$

$$C^{(2)} = \begin{array}{|cccc|cccc|cccc|}
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
8 & 9 & 10 & 11 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 0 & 1 & 2 & 3 \\
\hline
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 \\
10 & 11 & 8 & 9 & 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 & 2 & 3 & 0 & 1 \\
\hline
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 \\
11 & 10 & 9 & 8 & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 & 3 & 2 & 1 & 0 \\
\hline
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 \\
9 & 8 & 11 & 10 & 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 & 1 & 0 & 3 & 2 \\
\hline
\end{array}$$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ODSA 2010
September 13 – 15, 2010
University of Rostock, Germany
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Thank you!