Trellis Representations for Linear Block Codes

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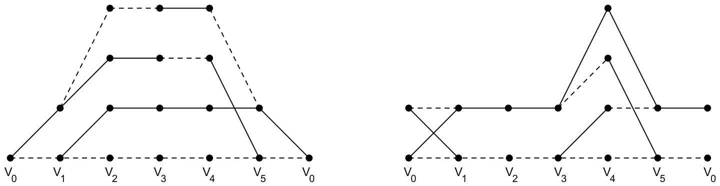
What is a Trellis Representation?

$$\mathcal{C} = \mathrm{im}\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \subseteq \mathbb{F}_2^6$$



- - - - label 0 ——— label 1

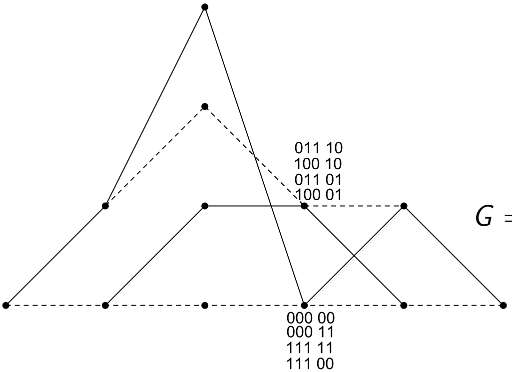
Code = set of edge-label sequences of all <u>cycles</u> through the graph.

Motivation:

- decoding by (variant of) Viterbi algorithm
- code structure

What is a Trellis Representation?

Example:



$$G = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Basic Notions of Trellis Representations

Definition

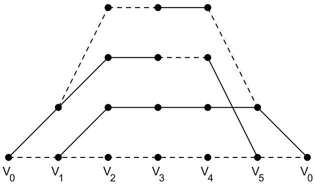
A **trellis** is a graph $T = (V, E)$, where $V = \bigcup_{i=0}^{n} V_i$, $E = \bigcup_{i=0}^{n} E_i$ such that

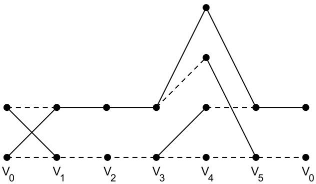
- $V_0 = V_n$,
- $E_i = \{v \xrightarrow{a} w \mid v \in V_i,\ w \in V_{i+1},\ a \in \mathbb{F}\}$ for $i = 0, \ldots, n-1$.

Edge-label code

$$\mathcal{C}(T) := \left\{ (c_0, \ldots, c_{n-1}) \in \mathbb{F}^n \;\middle|\; \begin{array}{l} \text{there exists a cycle in } T \\ v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \ldots \xrightarrow{c_{n-1}} v_n = v_0 \end{array} \right\}$$

- T **represents the code** $\mathcal{C} \subseteq \mathbb{F}^n$ if $\mathcal{C}(T) = \mathcal{C}$.
- T is called **conventional** if $|V_0| = 1$ and **tail-biting** else.





Further Notions

Linear Trellis

- each vertex and edge appears in a cycle,
- V_i is an $\mathbb{F}$-vector space for all i (after suitable labeling),
- The label code

$$\left\{ v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \dots \xrightarrow{c_{n-1}} v_n = v_0 \right\}$$

is a subspace of $V_0 \times \mathbb{F} \times V_1 \times \ldots \times \mathbb{F} \times V_{n-1} \times \mathbb{F}$.

Throughout this talk: only linear trellises!

Write $(v, a, w) \in V_i \times \mathbb{F} \times V_{i+1}$ for $v \xrightarrow{a} w$. Hence $E_i \subseteq V_i \times \mathbb{F} \times V_{i+1}$.

One-to-One Trellis

$$\mathcal{C}(T) \overset{\text{bijective}}{\longleftrightarrow} \text{cycles in } T.$$

Minimality and Non-Mergeability

Minimal Trellis

There exists no trellis $T' = (V', E')$ such that $\mathcal{C}(T') = \mathcal{C}$ and

$$|V_i'| \leq |V_i| \text{ for all } i \text{ and } |V_j'| < |V_j| \text{ for some } j.$$

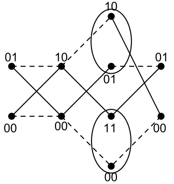
Mergeable Trellis

There exist distinct vertices $v, w \in V_i$ that can be **merged**, that is,

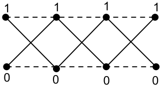
replacing v, w by a single vertex $\hat{v} \in V_i$
and all in- and outgoing edges of v, w accordingly

results in a trellis T' satisfying $\mathcal{C}(T) = \mathcal{C}(T')$.

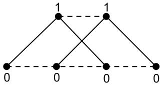
By linearity: Merging amounts to taking a certain quotient space of V_i.



one-to-one, mergeable



non-mergeable, not one-to-one
not minimal



non-mergeable, one-to-one
minimal

$$\mathcal{C}(T) = \{000,\ 110,\ 011,\ 101\}$$

How to Construct Minimal Trellises?

How to Construct Minimal Trellises?

Theorem (Forney '88, Muder '88, McEliece '92)

Let $T = (V, E)$ be a conventional trellis of $\mathcal{C}$. Then the following are **equivalent**:

- T is minimal (in the class of conventional trellises),
- T is non-mergeable,
- every conventional trellis T' of $\mathcal{C}$ can be merged to T, in particular,
$$|V_i| \leq |V_i'| \text{ for all } i = 0, \ldots, n-1,$$

The minimal conventional trellis of $\mathcal{C}$ is unique up to trellis isomorphism.

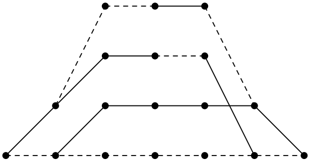
Forney's Construction:

$$V_i := \mathcal{C}/\mathcal{C}_i, \quad E_i := \left\{ \left([c]_i,\, c_i,\, [c]_{i+1} \right) \,\middle|\, c \in \mathcal{C} \right\},$$

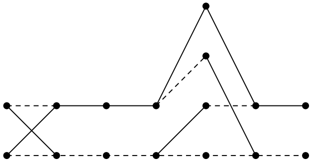
$$\begin{aligned} \mathcal{C}_i &:= \{(c_0, \ldots, c_{n-1}) \in \mathcal{C} \mid (c_0, \ldots, c_{i-1}, 0, \ldots, 0) \in \mathcal{C}\} \\ &= \{(c_0, \ldots, c_{n-1}) \in \mathcal{C} \mid (0, \ldots, 0, c_i, \ldots, c_{n-1}) \in \mathcal{C}\} \end{aligned}$$

Why Considering Tail-Biting Trellises?

$$\mathcal{C} = \operatorname{im} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \subseteq \mathbb{F}_2^6$$



17 vertices, 20 edges



14 vertices, 18 edges

Both trellises are minimal.

How to Construct Tail-Biting Trellises?

Example (Dimension 1): $\mathcal{C} = \mathrm{im}\,(1,\, 2,\, 0,\, 1,\, 1) \subseteq \mathbb{F}_3^5.$

Possible spans: $(0,\, 4]$, $(1,\, 0]$, $(3,\, 1]$, $(4,\, 3]$.

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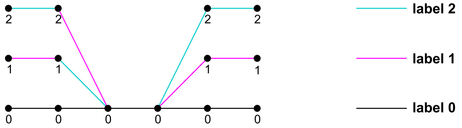
Choose the span $(3,\ 1]$ and put

$$V_i = \begin{Bmatrix} \mathbb{F}_3, & \text{if } i \in (3,\ 1] \\ \{0\}, & \text{else} \end{Bmatrix} = \mathrm{im}\, v_i, \quad \text{where } v_i = \begin{Bmatrix} 1, & \text{if } i \in (3,\ 1] \\ 0, & \text{else} \end{Bmatrix}$$

$$E_i = \mathrm{im}\,(v_i, c_i, v_{i+1}) = \{(\alpha v_i, \alpha c_i,\ \alpha v_{i+1}) \mid \alpha \in \mathbb{F}_3\}$$

This results in the one-to-one and minimal trellis



How to Construct Tail-Biting Trellises?

Theorem (Kschischang/Sorokine '95)

Let $T' = (V', E')$ and $T'' = (V'', E'')$ be trellises of $\mathcal{C}'$ and $\mathcal{C}''$. Define

$$V_i = V_i' \times V_i''$$

$$E_i = \left\{ \left((v,w), a+b, (\hat{v},\hat{w})\right) \,\middle|\, (v,a,w) \in E_i',\ (\hat{v},b,\hat{w}) \in E_i'' \right\}.$$

Then $T = (V, E)$ is a trellis of $\mathcal{C}' + \mathcal{C}''$.

If T' and T'' are one-to-one and $\mathcal{C}' \cap \mathcal{C}'' = \{0\}$, then T is one-to-one.

Product Trellis

Let $\mathcal{C} = \operatorname{im} G$ and $\mathcal{S}$ be a list of spans for the rows of G. Define

$$T_{G,\mathcal{S}}$$

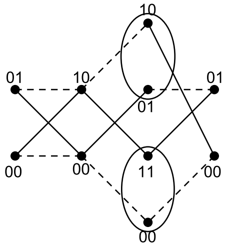
as the product of the corresponding 1-dimensional trellises.

Product trellises

- are linear and one-to-one,
- but may be mergeable and thus not minimal.

Example

$$\mathcal{C} = \mathrm{im}\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \subseteq \mathbb{F}_2^3, \quad \mathcal{S} = \begin{bmatrix}(0,\ 2]\\ (1,\ 0]\end{bmatrix}$$



The Minimal Conventional Trellis as a Product Trellis

Theorem (Kschischang/Sorokine '95, McEliece, '96)

There exists a pair $(G, \mathcal{S})$ such that the span list

$$\mathcal{S} = \left[(a_l,\ b_l],\ l = 1, \ldots, k\right]$$

satisfies

- $(a_l,\ b_l]$ is conventional for all $l = 1, \ldots, k$,
- $a_1, \ldots, a_k$ are distinct,
- $b_1, \ldots, b_k$ are distinct.

The corresponding product trellis $T_{G,\mathcal{S}}$ is the minimal conventional trellis of $\mathcal{C} = \operatorname{im} G$.

The span list $\mathcal{S}$ is uniquely determined by $\mathcal{C}$.

We call G a **conventional trellis-oriented generator matrix** of $\mathcal{C}$.

Characteristic Pair of a Code

$\mathcal{C} \subseteq \mathbb{F}^n$ be a k-dimensional code with support $\{0, \dots, n-1\}$.

Theorem (generalized version of Koetter/Vardy, 2003)

There exists a **characteristic pair** of $\mathcal{C}$, that is,

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^{n \times n} \text{ and } \mathcal{T} = \begin{bmatrix} (a_1, b_1] \\ \vdots \\ (a_n, b_n] \end{bmatrix}$$

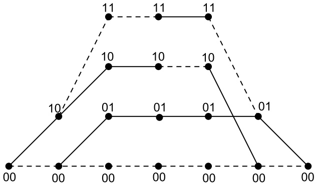
with the following properties

- $\operatorname{im} X = \mathcal{C}$, that is, $\{x_1, \dots, x_n\}$ forms a generating set of $\mathcal{C}$.
- $(a_l, b_l]$ is a span of x_l for $l = 1, \dots, n$.
- $a_1, \dots, a_n$ are distinct and $b_1, \dots, b_n$ are distinct.
- For all $j = 0, \dots, n-1$ the shifted pair $(\sigma^j(X), \sigma^j(\mathcal{T}))$ contains a conventional trellis-oriented generator matrix of $\sigma^j(\mathcal{C})$.

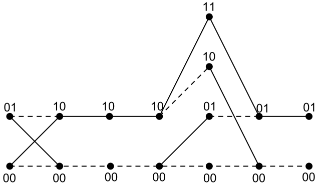
The span list $\mathcal{T}$ is uniquely determined by $\mathcal{C}$, the matrix X is not.

Example: $\mathcal{C} = \operatorname{im}\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. Then

$$X = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad \mathcal{T} = \begin{bmatrix} (0,\ 4] \\ (1,\ 5] \\ (3,\ 0] \\ (2,\ 1] \\ (4,\ 2] \\ (5,\ 3] \end{bmatrix}$$



$$\mathcal{S} = \big[(0,\ 4],\ (1,\ 5]\big]$$



$$\hat{\mathcal{S}} = \big[(0,\ 4],\ (3,\ 0]\big]$$

KV-Trellises

Definition

A **KV-trellis** of $\mathcal{C}$ is a product trellis $T_{G,\mathcal{S}}$, where

- $G \in \mathbb{F}^{k \times n}$ is a full row rank submatrix of a characteristic matrix of $\mathcal{C}$,
- $\mathcal{S}$ is the corresponding span list.

Theorem (Koetter/Vardy, 2003)

Every minimal trellis is a KV-trellis (based on a suitable choice of the characteristic matrix). But not every KV-trellis is minimal.

Theorem (G_L/Weaver, 2010)

KV-trellises are non-mergeable.

For the proof ...

BCJR-Construction

...conventional trellises by Bahl, Cocke, Jelinek, Raviv (1974).

Definition (Nori/Shankar, 2006)

Let $\mathcal{C} = \operatorname{im} G = \ker H^{\mathsf{T}}$, where

$$G = (G_0, \ldots, G_{n-1}) \text{ and } H = (H_0, \ldots, H_{n-1}).$$

Choose $N_0 \in \mathbb{F}^{k \times (n-k)}$ and define $N_{i+1} = N_i + G_i H_i^{\mathsf{T}}$.
Then the trellis $T_{(G,H,N_0)}$ having vertex and edge spaces

$$V_i = \operatorname{im} N_i, \quad E_i = \operatorname{im}(N_i, G_i, N_{i+1}) = \left\{(\alpha N_i,\, \alpha G_i,\, \alpha N_{i+1}) \,\middle|\, \alpha \in \mathbb{F}^k\right\}$$

is linear and represents the code $\mathcal{C}$.

- N_0 is a design parameter.
- $N_0 = 0$ leads to the minimal conventional trellis.
- $T_{(G,H,N_0)}$ may be mergeable and not one-to-one.

BCJR-Construction

Theorem (G$_\text{L}$/Weaver, 2010)

Let $\mathcal{C} = \operatorname{im} G$ and $\mathcal{S} = \big[(a_l,\ b_l], l = 1, \dots, k\big]$ be a span list of G. Define

$$N_0, \text{ based on span list } \mathcal{S} \quad \text{(can be made precise)}.$$

Then

- $T_{(G,H,N_0)}$ is non-mergeable.
- The product trellis $T_{G,\mathcal{S}}$ can be merged to $T_{(G,H,N_0)}$.
- KV-trellises $T_{G,\mathcal{S}}$ are isomorphic to their counterpart $T_{(G,H,N_0)}$ and thus KV-trellises are non-mergeable.

But:

- BCJR-trellises may not be one-to-one.
- Not every one-to-one BCJR-trellises is a KV-trellises.

Future Work: Dual Trellises for $\mathcal{C}^\perp$

- A BCJR-trellis $T_{(G,H,N_0)}$ naturally gives rise to a dual trellis $T_{(H,G,N_0^\mathsf{T})}$ representing $\mathcal{C}^\perp$.
- But the dual trellis may be mergeable.
- Koetter/Vardy's characteristic pairs give rise to a

Conjecture about KV-trellises of $\mathcal{C}^\perp$

(Koetter/Vardy, 2003).

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  Conjecture about KV-trellises of $\mathcal{C}^\perp$

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Theorem ($\mathrm{G_L}$/Weaver, 2010)

Conjecture is true for <u>minimal</u> KV-trellises and in this case the KV-dual coincides with the BCJR-dual.

Tools:

- BCJR-dualization,
- dualizing the edge spaces (Forney, 2001).