On maximal partial spreads of the hermitian variety $H(3, q^2)$

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A geometry associated with a sesquilinear or quadratic form.

- the set of elements of the geometry is the set of all totally isotropic subspaces (or totally singular) of $V(n + 1, q)$ with relation to the form
- incidence is symmetrized containment
- The rank of the polar space is the Witt index of the form.
Finite classical polar spaces

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A finite generalized quadrangle (GQ) is a point-line geometry $S = (\mathcal{P}, \mathcal{B}, I)$ such that

(i) Each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point.

(iii) If $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x I M I y I L$. 
Finite classical GQs: associated to sesquilinear or quadratic forms of Witt index two.

- $Q^{-}(5, q)$: set of points of $\text{PG}(5, q)$ satisfying
  \[ g(X_0, X_1) + X_2 X_3 + X_4 X_5 = 0 \]
  where $g(X_0, X_1)$ is an irreducible homogenous polynomial of degree two.

- $H(3, q^2)$: set of points of $\text{PG}(3, q^2)$ satisfying
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Some properties

- $Q^-(5, q)$: order $(q, q^2)$
- $H(3, q^2)$: order $(q^2, q)$
- $Q(4, q)$: order $q$ (meaning: $(q, q)$).

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An *ovoid* of a GQ $S$ is a set $O$ of points of $S$ such that every line of $S$ contains exactly one point of $O$.

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Partial ovoids and partial spreads

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A *partial ovoid* of a GQ $S$ is a set $\mathcal{O}$ of points of $S$ such that every line of $S$ contains at most one point of $S$. A partial ovoid is *maximal* if it cannot be extended to a larger partial ovoid.

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Lemma

If $S$ is a GQ of order $(s, t)$, then an ovoid of $S$ has size $st + 1$, and a spread of $S$ has size $st + 1$
Theorem

$Q^{-}(5, q)$ has no ovoids

Corollary

$H(3, q^2)$ has no spreads
Theorem
$Q^{-}(5, q)$ has no ovoids

Corollary
$\mathbb{H}(3, q^2)$ has no spreads
An upper bound on the size

Theorem (DB, Klein, Metsch, Storme)

A partial spread of $H(3, q^2)$ has size at most $\frac{q^3 + q + 2}{2}$. 
\[ |\mathcal{B}| = q^3 + 1 - \delta, \ h = \delta(q^2 + 1) \]

Compute the number of triples in the set

\[ \{(S_1, S_2, P) \mid S_1, S_2 \in \mathcal{B}, P \in S\} \]

where the unique projective line on \( P \) meeting \( S_1 \) and \( S_2 \) is a line of \( S \).

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lower bound for the number of elements in the set

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Partial spreads of $\mathbb{H}(3, q^2)$

Examples of size $O(q^2)$

Let $S_1, S_2 \in B$ such that the number of triples $(S_1, S_2, P)$ is maximal (denote this number $\alpha$). Use the lower bound to define $\alpha_0$

\[ |B|(|B| - 1)\alpha_0 := \delta(q^2 + 1)|B| \left( \frac{|B|}{q+1} - 1 \right) \]

- it follows that $\alpha \geq \alpha_0$
- For any two $S_1, S_2 \in B$ there are $(q^2 + 1)(q^2 - 1)$ candidates to be a hole.
- Any $S \in B \setminus \{S_1, S_2\}$ kills $q + 1$ candidates, but at least $\alpha_0$ of these candidates are holes

\[ (|B| - 2)(q + 1) + \alpha_0 \leq q^4 - 1 \]

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Theorem (DB, Klein, Metsch, Storme (2008))

A partial spread of $H(3, q^2)$ has size at most $\frac{q^3 + q + 2}{2}$. 
Examples for $q = 2, 3$

**Theorem (Dye)**

*There exists a maximal partial ovoid of $Q^{-}(5, 2)$ of size 6.*

**Theorem (Ebert and Hirschfeld)**

*There exists a maximal partial spread of $H(3, 9)$ of size 16*

**Theorem (Cossidente)**

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When equality holds

Corollary

If $H(3, q^2)$ has a spread of size $\frac{q^3 + q + 2}{2}$, then there exists a symmetric $2 - (v, k, \lambda)$ design, with $v = \frac{q^3 + q + 2}{2}$, $k = q^2 + 1$, $\lambda = 2q$. 
The case $q = 4$

Exhaustive search:

- no maximal partial spread exist with size in the interval $[26, \ldots, 35]$,
- we found all maximal partial spreads with size in $\{23, 24, 25\}$
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maximal partial spreads of size \((q + 1)^2\)

\(H(3, q^2)\) has maximal partial spreads of size \((q + 1)^2\) for

- \(q = 2^{2h}, h \geq 1\).
- \(q = 3 \pmod{4}\)
- \(q = 9\)
The case $q = 5$

In this case we searched for maximal partial ovoids of $Q^-(5, q)$.

- Exhaustive search: no maximal partial ovoid exist with size in the interval $[49, \ldots, 66]$,
- we found a maximal partial ovoid of size 48,
- exhaustive search: we found all maximal partial ovoids containing a conic with size in $\{40, 41, 42, 43\}$,
- exhaustive search: we found no maximal partial ovoids containing a conic with size in $\{44, 45, 46, 47\}$
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$H(3, q^2)$ has partial spreads of size $q + 1 + 3 \frac{q^2 - q}{2}$ (by a construction of Thas).

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### An overview

<table>
<thead>
<tr>
<th>TUB: $\frac{q^3+q+2}{2}$</th>
<th>$(q + 1)^2$</th>
<th>$q + 1 + 3\frac{q^2-q}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 3$</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>$q = 4$</td>
<td>35</td>
<td>25</td>
</tr>
<tr>
<td>$q = 5$</td>
<td>66$^1$</td>
<td>36</td>
</tr>
<tr>
<td>$q = 7$</td>
<td>176$^2$</td>
<td>64</td>
</tr>
</tbody>
</table>

$^1$not reached  
$^2$open
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For $q = 5$, two of them can be glued together to produce the maximal partial ovoid of size 48 of $Q^-(5, q)$.

This is *not* possible for $q = 7$ …… but it is possible for $q = 11$. 
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The case $q = 7$ and beyond

- $q = 7$: examples of size 96 and 98 (Cimrakova, Coolsaet)
- $q = 11$: example of size 240 different from glued example (Coolsaet)
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