Construction of q-analogs of combinatorial designs

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q-Analogs of Combinatorial Designs

Definition

q-analog of $t-(n,k,\lambda)$-design

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$$\mathcal{B} \subseteq \begin{bmatrix} GF(q)^n \\ k \end{bmatrix}_q : |\{K \in \mathcal{B} \mid T \leq K\}| = \lambda \quad \forall\ T \in \begin{bmatrix} GF(q)^n \\ t \end{bmatrix}_q$$

History of Designs over Finite Fields

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- M. Braun (2005):

  $$3-(8,4,11,2)\text{-design}$$

q-Steiner Systems and Network Codes

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A $t-(n, k, 1; q)$*-design is called a q-Steiner system.*

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- Error-correcting network code = a set of k-subspaces in $GF(q)^n$ such that each t-subspace is in at most 1 k-subspace
- Perfect code = a set of k-subspaces in $GF(q)^n$ such that each t-subspace is in exactly 1 k-subspace

Construction

$\mathcal{M}$:= incidence matrix between k-subspaces and t-subspaces of $GF(q)^n$

$$\mathcal{M}_{T,K} := \begin{cases} 1 & \text{if } t\text{-subspace } T \leq k\text{-subspace } K \\ 0 & \text{else} \end{cases} .$$

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Solve the diophantine system of equations

$$\mathcal{M} \cdot \vec{x} = \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix}$$

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PROBLEM: Size of $\mathcal{M}$ grows too fast for increasing parameters!

Construction – Kramer-Mesner method

Prescribing a group G of automorphisms of the design reduces the size of $\mathcal{M}$

$\Rightarrow$ shrinked Kramer-Mesner matrix $\mathcal{M}^G :=$ incidence matrix between the G-orbits of k-subspaces and the G-orbits of t-subspaces of $GF(q)^n$

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Existing Implementation

Implementation with Double Cosets for the construction of $G \backslash\!\backslash \begin{bmatrix} GF(q)^n \\ k \end{bmatrix}_q$

Transform the problem of constructing $G \backslash\!\backslash \begin{bmatrix} GF(q)^n \\ k \end{bmatrix}_q$ into a double coset problem:

$$G \backslash\!\backslash \begin{bmatrix} GF(q)^n \\ k \end{bmatrix}_q \twoheadrightarrow G \backslash GL(n,q) / GL(n,q)_{\langle e_1,\ldots,e_k \rangle}$$

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PROBLEM: Works just a for a few selected groups

New Implementation

- Schreier-Sims algorithm for $G \leq GL(n, q)$
- Direct construction of $G \backslash\!\backslash \begin{bmatrix} GF(q)^n \\ k \end{bmatrix}_q$ via the laddergame

Schreier-Sims Algorithm for Matrix Groups

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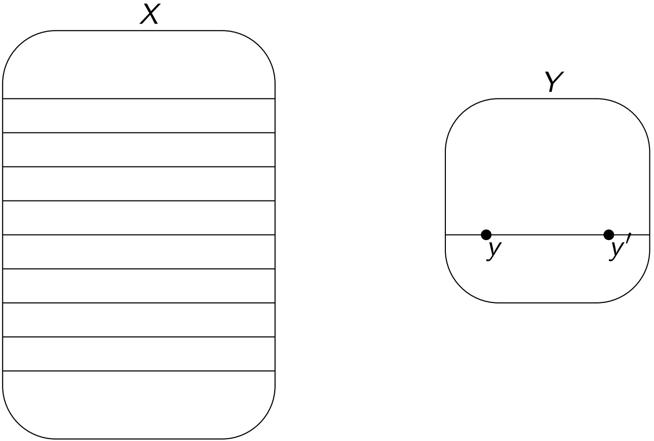
$$T_1 \geq T_2 \geq \cdots \geq T_n\ , \quad T_i \in \mathcal{T}(G_i/G_{i+1})$$

$\Rightarrow$ $T_{i_{(i=1,\dots,n)}}$ as Input for Construction of $G \backslash\!\!\backslash \begin{bmatrix} GF(q)^n \\ k \end{bmatrix}_q$

Homomorphism Principle

$\varphi : X \to Y$ is a surjective G-homomorphism



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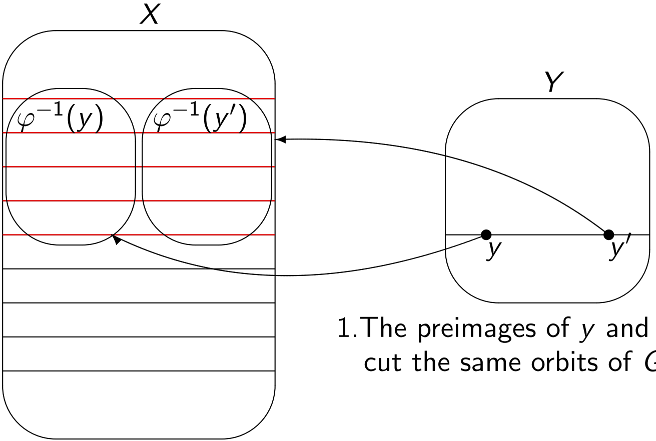

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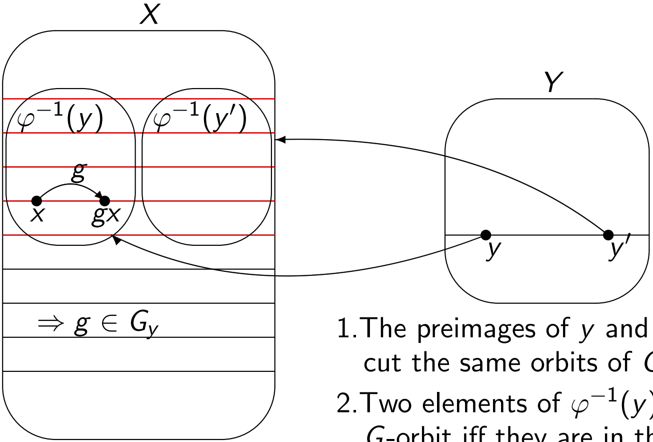
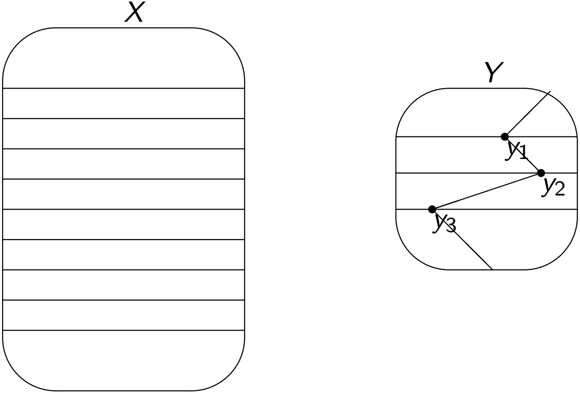


1. The preimages of y and y' cut the same orbits of G in X
2. Two elements of $\varphi^{-1}(y)$ are in the same G-orbit iff they are in the same orbit under G_y

1.case: get $G \backslash\backslash X$ from $G \backslash\backslash Y$ by splitting orbits



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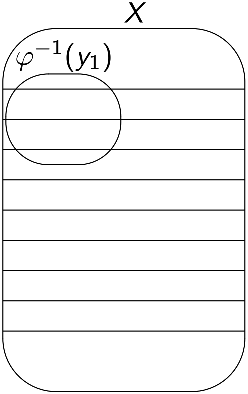

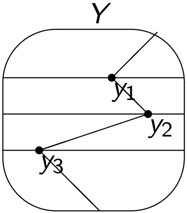

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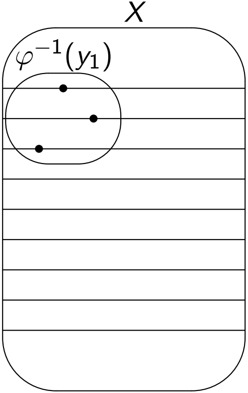

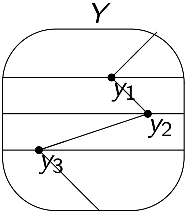

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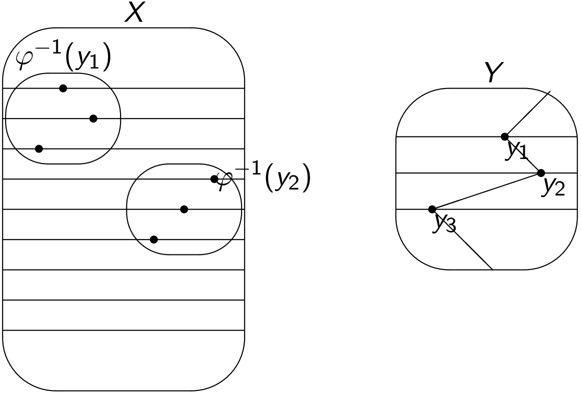

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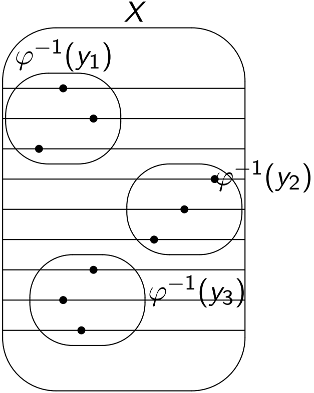

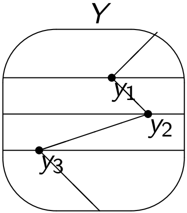

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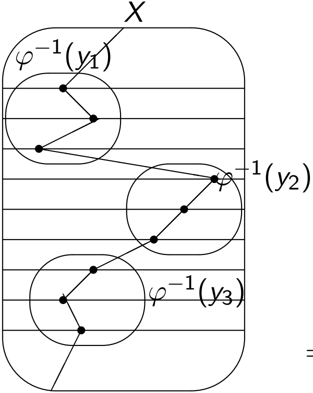

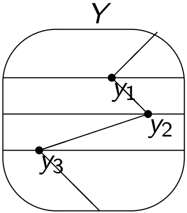

$$\Rightarrow \bigcup_i (G_{y_i} \backslash\backslash \varphi^{-1}(y_i)) \in \mathcal{T}(G \backslash\backslash X)$$

2.case: get $G \backslash\backslash Y$ from $G \backslash\backslash X$ by fusing orbits

Laddergame

$Y_i := \{y \leq GF(q)^n \mid dim(y) = i\}$

$X_i := \{(y, t) \mid y \in Y_{i-1}, t \in Y_1, t \nsubseteq y\}$

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$Y_i := \{y \leq GF(q)^n \mid dim(y) = i\}$

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- Downstep – Splitting orbits

$$\varphi_i : X_i \longrightarrow Y_{i-1},\ (y, t) \mapsto y$$

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- Upstep – Fusing orbits

$$\delta_i : X_i \to Y_i,\, (y, t) \mapsto \langle y \cup t \rangle$$

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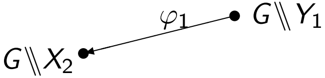
- $G \mathbin{\!/\mkern-5mu/\!} Y_1$

Laddergame



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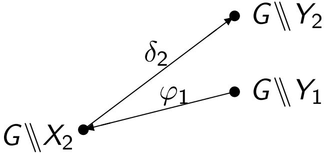

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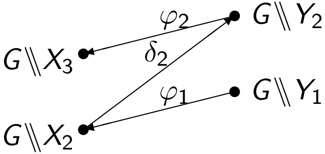

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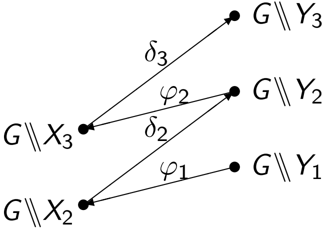

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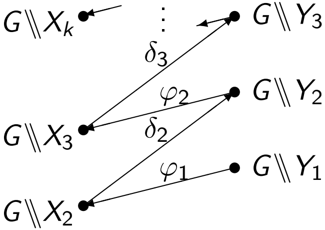

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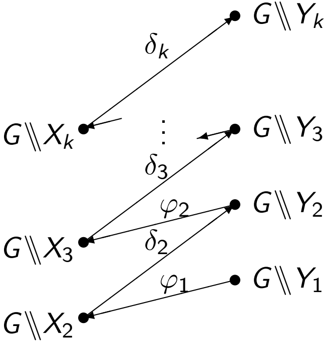

$G \mathbin{/\!/} Y_k$
δ_k
$G \mathbin{/\!/} X_k$
$G \mathbin{/\!/} Y_3$
δ_3
φ_2
$G \mathbin{/\!/} Y_2$
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δ_2
φ_1
$G \mathbin{/\!/} Y_1$
$G \mathbin{/\!/} X_2$

New Results

parameters	$\|G\|$	dim $\mathcal{M}_{t,k}^{G}$	λ
$2-(6,3,\lambda;3)$	336	93×234	16
$2-(8,4,\lambda;2)$	1020	15×217	$35, 56, 70, 105, 126, 161,$ $176, 196, 245, 266, 280, 315$
$2-(9,3,\lambda;2)$	1533	31×529	$21, 22, 42, 43, 63$
$2-(9,4,\lambda;2)$	4599	11×725	$21, 63, 84, 126, 147, 189, 210,$ $252, 273, 315, 336, 378, 399, 462$ $504, 525, 567, 588, 651, 693$ $714, 756, 777, 840, 882, 903$ $945, 966, 1008, 1029, 1071, 1092$ $1134, 1155, 1197, 1218, 1281, 1323$

Open Problems

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- Designs with $t > 3$?

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Thank you very much for your attention!

References

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