Regular antichains

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Content

Introduction

Definitions

Definition (Antichain)

Let $\mathcal{B}$ be a subset of $2^{[m]}$ with $[m] := \{1, 2, \ldots, m\}$.
We call $\mathcal{B}$ an antichain (AC) if there are no two sets in $\mathcal{B}$ which are comparable under set inclusion.

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Example

$\mathcal{B} := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \subseteq 2^{[3]}$.

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$\mathcal{B} := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \subseteq 2^{[3]}$.

The size of $\mathcal{B}$ is $n := |\mathcal{B}|$.

Definitions

Definition (k-regular)

An antichain $\mathcal{B}$ is called k-regular ($k \in \mathbb{N}$), if for each $i \in [m]$ there are exactly k blocks $B_1, B_2, \ldots, B_k \in \mathcal{B}$ containing i.

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Example

$\mathcal{B} := \{\{1,2\},\{1,3\},\{2,3\}\}$ is a 2-regular antichain.

Examples for (k, m, n)-antichains

Example

1. $(1, 1, 1)$-antichain:
   $\{\{1\}\}$
2. $(2, 5, 5)$-antichain:
   $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$
3. $(3, 6, 6)$-antichain:
   $\{\{1, 2\}, \{1, 4, 6\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5, 6\}\}$

Introduction to the problem

If $\mathcal{B} \subseteq 2^{[m]}$ is a k-regular antichain of size n we call $\mathcal{B}$ a (k, m, n)-antichain.

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Problem

Decide for given parameters k, m, n if there exists a (k, m, n)-antichain or not.

Dual structure

Definition (separable)

Let $\mathcal{C}$ be a collection of sets. The point x is separable from y if there exists a set $C \in \mathcal{C}$ such that $x \in C$ and $y \notin C$.

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The collection of blocks

$$\mathcal{C} = \{\{a\}, \{a, b\}, \{c, d\}\}$$

separates a from b but not b from a.

Dual structure

Definition (Completely Separating System)

A Completely Separating System (CSS) $\mathcal{C}$ on $[n]$ is a collection of blocks of $[n]$ in which for any set $\{x, y\}$ of distinct points $x, y \in [n]$, x and y are separable from each other.

Dual structure

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The collection of blocks

$$\mathcal{C} = \{\{1, 2, 3, 4\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{6\}\}$$

is a CSS on six points.

Necessary conditions of existence

Volume of a regular antichain

Definition (Volume)

Let $\mathcal{B} \subseteq 2^{[m]}$. The volume of $\mathcal{B}$ is

$$v(\mathcal{B}) := \sum_{B \in \mathcal{B}} |B|.$$

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Lemma

Let $\mathcal{B}$ be a (k, m, n)-AC. Then

$$v(\mathcal{B}) = km.$$

Lower and upper bounds for n

Lemma

Let $\mathcal{B}$ be a (k, m, n)-AC with $k > 1$. Then

$$k + 1 \leq n \leq \frac{km}{2}.$$

Lower and upper bounds for n

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$$k + 1 \leq n \leq \frac{km}{2}.$$

Proof

$$km = v(\mathcal{B}) = \sum_{B \in \mathcal{B}} |B| \quad \left\{ \begin{array}{lr} \leq & (m-1)n \\ \geq & 2n \end{array} \right. .$$

□

Lower and upper bounds for n

Lemma

Let $\mathcal{B}$ be an (m, m, n)-AC with $m \geq 6$, then

$$m + 3 \leq n < \binom{m}{2} - \frac{m}{3}.$$

Lower and upper bounds for n

Lemma

Let $\mathcal{B}$ be an (m, m, n)-AC with $m \geq 6$, then

$$m + 3 \leq n < \binom{m}{2} - \frac{m}{3}.$$

- For all $B \in \mathcal{B}$ is $|B| \leq m - 2$.
- Every point is in at least five 3-sets.

Recursive constructions

Recursive Construction (1) - Add one point

Lemma

Let $\mathcal{B}$ be a (k, m, n)-AC. Then there exists a $(k, m+1, n)$-AC $\mathcal{B}'$.

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Example

$\mathcal{B} := \{123, 14, 245, 35\}$
$\mathcal{B}' := \{123, 14, 2456, 356\}$

Consequence

Definition (R(n,k))

Let n, k be natural numbers. If there is a k-regular antichain of size n then $R := R(n, k)$ is the smallest natural number such that a (k, R, n)-antichain exists. That means:

$$R(n, k) := \min\{m : \exists (k, m, n)\text{-antichain}\}.$$

Complementary antichain

Definition (complement)

Let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be an antichain on $[m]$. The complement $\overline{\mathcal{B}}$ of $\mathcal{B}$ is defined by $\overline{\mathcal{B}} = \{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_n\}$ with $\overline{B}_i := [m] \backslash B_i$.

Complementary antichain

Example

$\{12345, 126, 37, 47, 56\}$

	B_1	B_2	B_3	B_4	B_5
1	1	1			
2	1	1			
3	1		1		
4	1			1	
5	1				1
6		1			1
7			1	1	

Complementary antichain

Example

$\{1234567, 1234567, 1234567, 1234567, 1234567\}$

	B_1	B_2	B_3	B_4	B_5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1

Complementary antichain

Example

$\{67, 3457, 12456, 12356, 12347\}$

	$\overline{B}_1$	$\overline{B}_2$	$\overline{B}_3$	$\overline{B}_4$	$\overline{B}_5$
1			1	1	1
2			1	1	1
3		1		1	1
4		1	1		1
5		1	1	1	
6	1		1	1	
7	1	1			1

Complementary antichain - Consequence

Lemma

Let $\mathcal{B}$ be a (k, m, n)-AC. Then there exists an $(n - k, m, n)$-AC.

Complementary antichain - Consequence

Lemma

Let $\mathcal{B}$ be a (k, m, n)-AC. Then there exists an $(n-k, m, n)$-AC.

Remark

It is sufficient to be only interested in the cases $k \leq \frac{n}{2}$.

Values of $R(n, k)$ for $2 \leq n \leq 31$ and $k \leq 12$

n	1	2	3	4	5	6	7	8	9	10	11	12
2	2											
3	3											
4	4	4										
5	5	5										
6	6	6	4									
7	7	7	5									
8	8	8	6	5								
9	9	9	6	6								
10	10	10	7	5	6							
11	11	11	8	6	6							
12	12	12	8	6	6	6						
13	13	13	9	7	6	7						
14	14	14	10	7	7	7	6					
15	15	15	10	8	6	7	7					
16	16	16	11	8	7	7	7	6				
17	17	17	12	9	7	7	7	7				
18	18	18	12	9	8	7	8	7	6			
19	19	19	13	10	8	7	8	7	7			
20	20	20	14	10	8	8	8	8	7	6		
21	21	21	14	11	9	7	8	8	7	7		
22	22	22	15	11	9	8	8	8	8	7	7	
23	23	23	16	12	10	8	8	8	8	7	7	
24	24	24	16	12	10	8	8	8	8	8	7	7
25	25	25	17	13	10	9	8	9	8	8	7	7
26	26	26	18	13	11	9	8	9	8	8	7	8
27	27	27	18	14	11	9	9	9	9	8	7	8
28	28	28	19	14	12	10	8	9	9	8	8	7
29	29	29	20	15	12	10	9	9	9	9	8	8
30	30	30	20	15	12	10	9	9	9	9	8	8
31	31	31	21	16	13	11	9	9	9	9	8	8

Recursive construction (2)

Theorem

Let $\mathcal{B}$ be an (m, m, n)-AC. If there exists an $i \in \mathbb{N}$ such that $|B| > i \geq 2$ for all $B \in \mathcal{B}$, then there exists an $(m+i, m+i, n+m)$-AC $\mathcal{B}'$.

Recursive construction (2) - Example

Example ($(7, 7, 12)$-antichain $\mapsto$ $(9, 9, 19)$-antichain)

1	1	1			1	1		1		1	1		
2	1	1	1			1	1		1	1			
3	1	1	1	1			1	1			1		
4	1		1	1	1			1	1			1	
5	1			1	1	1	1		1			1	
6		1	1	1	1					1	1	1	
7						1	1	1	1	1	1	1	
8													
9													

Recursive construction (2) - Example

Example ($(7,7,12)$-antichain $\mapsto$ $(9,9,19)$-antichain)

1	1	1			1	1		1		1	1		1						1
2	1	1	1			1	1		1	1			1	1					
3	1	1	1	1			1	1			1			1	1				
4	1		1	1	1			1	1			1			1	1			
5	1			1	1	1	1		1			1				1	1		
6		1	1	1	1					1	1	1					1	1	
7						1	1	1	1	1	1	1						1	1
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1	1	1			1	1		1		1	1		1						1
2	1	1	1			1	1		1	1			1	1					
3	1	1	1	1			1	1			1			1	1				
4	1		1	1	1			1	1			1			1	1			
5	1			1	1	1	1		1			1				1	1		
6		1	1	1	1					1	1	1					1	1	
7						1	1	1	1	1	1	1						1	1
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1	1	1			1	1		1		1	1		1						1
2	1	1	1			1	1		1	1			1	1					
3	1	1	1	1			1	1			1			1	1				
4	1		1	1	1			1	1			1			1	1			
5	1			1	1	1	1		1			1				1	1		
6		1	1	1	1					1	1	1					1	1	
7						1	1	1	1	1	1	1						1	1
8							1				1		1	1	1	1	1	1	1
9		1						1					1	1	1	1	1	1	1

Recursive construction (3)

Theorem

Let $\mathcal{B}_i$ *be a* (k_i, m_i, n_i)*-AC with* $B \in \mathcal{B}_i \Rightarrow |B| > 1$ $(i = 1, 2)$. *If* $k_1 + m_2 = m_1 + k_2$ *then there exists a* $(k_1 + m_2, m_1 + m_2, n_1 + n_2 + m_1 m_2)$*-AC* $\mathcal{B}'$.

Recursive construction (3)

Theorem

Let $\mathcal{B}_i$ be a (k_i, m_i, n_i)-AC with $B \in \mathcal{B}_i \Rightarrow |B| > 1$ $(i = 1, 2)$. If $k_1 + m_2 = m_1 + k_2$ then there exists a $(k_1 + m_2, m_1 + m_2, n_1 + n_2 + m_1 m_2)$-AC $\mathcal{B}'$.

Example

$\mathcal{B}_1 = \{123, 14, 234\}$
$\mathcal{B}_2 = \{567\}$
$\mathcal{B}' = \{123, 14, 234, 567, 15, 16, 17, 25, 26, 27, 35, 36, 37, 45, 46, 47, \}$

Recursive construction (3)

Theorem

Let $\mathcal{B}_i$ be a (k_i, m_i, n_i)-AC with $B \in \mathcal{B}_i \Rightarrow |B| > 1$ $(i = 1, 2)$. If $k_1 + m_2 = m_1 + k_2$ then there exists a $(k_1 + m_2, m_1 + m_2, n_1 + n_2 + m_1 m_2)$-AC $\mathcal{B}'$.

Example

$\mathcal{B}_1 = \{123, 14, 234\}$
$\mathcal{B}_2 = \{567\}$
$\mathcal{B}' = \{123, 14, 234, 567, 15, 16, 17, 25, 26, 27, 35, 36, 37, 45, 46, 47, \}$

Proof

$\mathcal{B}' := \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{\{i, j\} : i \in [m_1], j \in [m_1 + 1, m_1 + m_2]\}$. □

Sufficient conditions of existence

Main Results

Theorem

Let $m, n \in \mathbb{N}$ *with* $m \geq 6$ *and with*

$$\begin{cases} m+3 \leq n \leq \left\lfloor \binom{m}{2} - \frac{2}{5}m \right\rfloor & \text{if} \quad m \equiv 0,1,3,4 \mod 5, \\ m+3 \leq n \leq \left\lfloor \binom{m}{2} - \frac{2}{5}m \right\rfloor - 1 & \text{if} \quad m \equiv 2 \qquad \mod 5. \end{cases}$$

Then there exists an (m, m, n)*-AC* $\mathcal{B}$*.*

Main Results

Theorem

Let $m \geq 3$ *be a natural number.*
An $(m-1, m, n)$*-antichain exists if and only if*

$$n \in \left[m+2, \binom{m}{2} - 2\right] \cup \left\{m, \binom{m}{2}\right\}.$$

Main Results

Theorem

Let $k, m \in \mathbb{N}$ *with* $2 \leq k \leq m - 2$.
Then a (k, m, n)*-AC exists if and only if*

$$k + 1 \leq n \leq \frac{km}{2}.$$

Conjecture

The Flat Antichain Theorem

Definition (flat)

An antichain is called flat if $||B| - |C|| \leq 1$ for all $B, C \in \mathcal{B}$.

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Theorem (Kisvölcsey, Lieby)

For every antichain $\mathcal{B} \subseteq 2^{[m]}$ there is a flat antichain $\mathcal{B}'$ with

$$|\mathcal{B}'| = |\mathcal{B}| \qquad \text{and} \qquad v(\mathcal{B}') = v(\mathcal{B}).$$

Conjecture

Conjecture

Let $k, m, n \in \mathbb{N}$ and $\mathcal{B}$ be a (k, m, n)-AC. Then there also exists a flat (k, m, n)-AC.

Thanks for your attention!