A Hilton-Milner theorem for vector spaces and the chromatic number of q-Kneser graphs

Aart B., A.E. Brouwer, A. Chowdhuri, P. Frankl,
T. Mussche, B. Patkós,T. Szőnyi

ALCOMA 10, April 17th 2010, Thurnau,
Organizers: thank you for this magnificent meeting,
Reinhard: Herzliche Glückwünsche zum 65. Geburtstag.

Who are we (and why so many)?

ab Aart Blokhuis, TU-Eindhoven, Netherlands (me)

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What do we look like?

Here are four of the authors (ab, aeb, tm and tsz) and some non-authors (áb, jat, tp, msz):

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A classical result in extremal combinatorics

Erdős-Ko-Rado, 61: If $\mathcal{F}$ is a k-uniform intersecting family of subsets of an n element set S, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ provided $2k \leq n$.

If $2k + 1 \leq n$, then equality holds if and only if $\mathcal{F}$ is the family of all subsets containing a fixed element $s \in S$.

The proof from the book, by G.O.H. Katona:
Arrange the points of S in a circle, and count how many members of $\mathcal{F}$ occupy a (k-)segment in this arrangement. Out of the n segments at most k belong to $\mathcal{F}$, and $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$.

A corresponding stability result

In case of equality in the Erdős-Ko-Rado theorem there is a point belonging to all sets, in other words, the covering number $\tau(\mathcal{F}) = 1$. What if $\tau \geq 2$?

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Theorem (Hilton-Milner, 67)

Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family with $2k+1 \leq n$ and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. The families achieving that size are

(i) *for a k-subset F and $x \notin F$ the family* $\{F\} \cup \{G \in \binom{[n]}{k} : x \in G, F \cap G \neq \emptyset\}$,

(ii) *if $k = 3$, then for any 3-subset S the family* $\{F \in \binom{[n]}{3} : |F \cap S| \geq 2\}$.

The q-analogue of the Erdős-Ko-Rado theorem

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$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1} \approx q^{k(n-k)}.$$

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Theorem (Hsieh, q-analogue of Erdős-Ko-Rado, 75)

$\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ intersecting, $n \geq 2k+1$, then $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. In case of equality we have a point pencil.

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Greene-Kleitman (78) did the case $n = 2k$ (not only point-pencils but also their duals!)

A challange!

Find a q-analogue of Katona's circle proof for Erdős-Ko-Rado, maybe using Singer-cycles, in any case using families of $q^n - 1$ k-spaces, with only $q^k - 1$ belonging to the family.

Gyula Katona (and Rudi Ahlswede)

Strongly intersecting families

Theorem (Frankl-Wilson, 86)

$\mathcal{F}$ a t-intersecting family of k-subspaces of an n-space, then:
$|\mathcal{F}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$ *if* $2k \leq n$,
and
$|\mathcal{F}| \leq \begin{bmatrix} 2k-t \\ k \end{bmatrix}$ *if* $2k - t \leq n \leq 2k$.

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Hilton-Milner families

A collection of k-spaces $\mathcal{F}$ is a *Hilton-Milner family* if

$$\mathcal{F} = \{U\} \cup \left\{ W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \leqslant W, \dim(W \cap U) \geq 1 \right\} \cup \begin{bmatrix} E \vee U \\ k \end{bmatrix},$$

for some fixed $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}, U \in \begin{bmatrix} V \\ k \end{bmatrix}$ with $E \not\leqslant U$.

The size of a H-M family is

$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k \quad \left(< \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}\right).$

Theorem (q-analogue of Hilton-Milner)

Let V be an n-dimensional vector space over $GF(q)$, $q \geq 3$ and $n \geq 2k+1$. Then for any intersecting family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ with $\tau(\mathcal{F}) \geq 2$ we have $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k$.

If $\mathcal{F}$ is of this size, then either $\mathcal{F}$ is a H-M family, or $k = 3$ and $\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : \dim(S \cap F) \geq 2\}$ for some 3-space S of V.

Furthermore if $k \geq 4$, then there exists an ε (independent of n, q, k) such that if $|\mathcal{F}| \geq (1-\varepsilon)\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k\right)$, then $\mathcal{F}$ is a subfamily of a H-M family.

Rough sketch of the proof:

Step 0: The case $k = 2$: Projectively we have an intersecting collection of lines. Now either they all pass through a point: E-K-R or they are in a plane: H-M.

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Key concept: a *hitting subspace*: meets all $F \in \mathcal{F}$.
$\tau(\mathcal{F})$: dimension of smallest hitting subspace.
A H-M familie has $\tau = 2$.
($\tau = 1$ iff point-pencil E-K-R)
$\tau = 2$: hitting lines (2-spaces),
$\mathcal{T}$: set of hitting lines ($\mathcal{F}$ is maximal $\rightarrow$ $\mathcal{T}$ intersecting)

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Step 1: $\tau = 2$, structure of $\mathcal{T}$ (see Step 0):
either $\begin{bmatrix} V \\ 2 \end{bmatrix}$, where $\dim V = 3$, or lines through a point (1-space) P in an $(l + 1)$-space.

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Step 2: Essential part of $\mathcal{F}$ is the k-spaces containing an element of $\mathcal{T}$.

Proposition (Description of $\mathcal{T}$-s if $\tau(\mathcal{F}) = 2$:)

Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Then

(i) $|\mathcal{T}| = 1$

(ii) $|\mathcal{T}| > 1$, $\tau(\mathcal{T}) = 1$, *and there is a* 2*-subspace* L *and a* 1*-subspace* $E \not\leqslant L$ *so that* $\mathcal{T} = \{\ell : \ell = E \vee W, \dim W = 1, W \leqslant L\}$.

(iii) $|\mathcal{T}| > 1$, $\tau(\mathcal{T}) = 1$, *and there is an* $l(\geq 3)$*-subspace* L *and a* 1*-subspace* $E \not\leqslant L$ *so that* $\mathcal{T} = \{\ell : \ell = E \vee W, \dim W = 1, W \leqslant L\}$.

(iv) $\tau(\mathcal{T}) = 2$, $\mathcal{T} = \begin{bmatrix} A \\ 2 \end{bmatrix}$ *for some* 3*-space* A *and* $\mathcal{F} = \{U : U \cap A \text{ has dimension } 2\}$ *and* $|\mathcal{F}| = (q^2 + q + 1)(\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - 1) + \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$.

In case (ii) and (iii) there is a 1*-subspace* E *and an* l*-subspace* L *such that* $\mathcal{F}$ *contains the set* $\mathcal{F}_{E,L}$ *of all* k*-spaces containing* E *and intersecting* L.

Proposition (Bounds for $\mathcal{F}$ in the previous prop.)

(i) $\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < |\mathcal{F}| < \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + (q+1)\left(\begin{bmatrix} k \\ 1 \end{bmatrix} - 1\right)\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$;

(ii) $(q+1)\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} \le |\mathcal{F}| \le$
$(q+1)\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} + (\begin{bmatrix} k \\ 1 \end{bmatrix})(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix})\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} + q^2\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$;

(iii) $\begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q\begin{bmatrix} l \\ 2 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} \le |\mathcal{F}| \le$
$\begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + (\begin{bmatrix} k \\ 1 \end{bmatrix})(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l \\ 1 \end{bmatrix})\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} + q^l\begin{bmatrix} n-l \\ k-l \end{bmatrix}$;

(iv) $|\mathcal{F}| = (q^2+q+1)(\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - 1) + \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$.

The last two terms of the upper bound for the size $\mathcal{F}$ given in (ii) and (iii) give an upper bound on $|\mathcal{F} \setminus \mathcal{F}_{E,L}|$.

Rough sketch of the proof (continued):

Step 3: Guarantee $\tau = 2$:

Proposition

Suppose that $k \geq 3$, $n \geq 2k+1$, *and for* $q = 2$ *we have* $n \geq 2k+2$. *Let* $|\mathcal{F}| \geq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} f(q)$. *For* $l \geq 3$ *we have the following: whenever an* l*-subspace is meeting every* $F \in \mathcal{F}$ *then there is an* $(l-1)$*-subspace meeting* $\mathcal{F}$ *if and only if*

$$\frac{q^{n-2k}(l-2)(q-1)^l f(q)}{(q^l-1)(q^k-1)} > 1. \tag{1}$$

Corollary

If $|\mathcal{F}| \geq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} \frac{q^2+q+1}{(q-1)^2} q^{3k-n}$, *then* $\tau(\mathcal{F}) = 2$, *that is* $\mathcal{F}$ *is contained in one of the systems described earlier.*

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Step 5: $k \geq 4$, $n = 2k + 1$, existence of hitting 3-spaces.

Corollary

If $|\mathcal{F}| \geq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} (1 - \frac{1}{q^3 - q})$, then $\tau(\mathcal{F}) \leq 2$ if $n \geq 2k + 2$ and $q \geq 3$, and $\tau(\mathcal{F}) \leq 3$ if $n = 2k + 1$ and $q \geq 4$.

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Step 6: $k \geq 4$, $n = 2k + 1$, $q \geq 4$ existence of hitting 3-spaces implies $\tau(\mathcal{F}) \leq 2$.

Idea: Averaging

$\exists E$, $\dim E = 1$ for which $|\mathcal{F}_E| \geq |\mathcal{F}| / \begin{bmatrix} k \\ 1 \end{bmatrix}$

If $\tau(\mathcal{F}) = \ell$, then $\exists E : |\mathcal{F}_E| \geq |\mathcal{F}| / \begin{bmatrix} \ell \\ 1 \end{bmatrix}$

$\tau(\mathcal{F}) > 1 \rightarrow \exists L \geq E$, $\dim L = 2$ for which

$|\mathcal{F}_L| \geq |F_E| / \begin{bmatrix} k \\ 1 \end{bmatrix} \geq |\mathcal{F}| / \begin{bmatrix} \ell \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}$

$\tau(\mathcal{F}) > 2 \rightarrow \exists W \geq L$, $\dim W = 3$ for which

$|\mathcal{F}_W| \geq |F_L| / \begin{bmatrix} k \\ 1 \end{bmatrix} \geq |\mathcal{F}| / \begin{bmatrix} \ell \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^2$

and so on...

Contradiction if the bound is larger than $\begin{bmatrix} n-s \\ k-s \end{bmatrix}$, or, in other words we found a hitting $(s-1)$-subspace.

For example,

If $A \notin \mathcal{T}$, $\dim A = 2$, then $|\mathcal{F}_A| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$.

Chromatic number of Kneser graphs

The vertex set of the Kneser graph $K_{n:k}$ is $\binom{V}{k}$, $|V| = n$. Two vertices (k-subsets) of $K_{n:k}$ are adjacent if they are disjoint.

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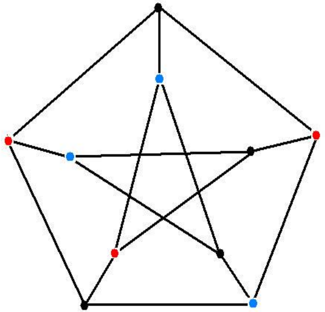

Lovász' theorem

Lovász 78: For $n = 2k + r$ the chromatic number of $K_{n:k}$ is $r + 2$.

The proof uses topological methods. A simpler proof was given by Bárány 78, a prize-winning proof by Joshua Greene 02 and recently an elementary (that is: not topological) proof by Matoušek 04.

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Chromatic number of the q-analogue of the Kneser graph

The vertex set of the q-Kneser graph $qK_{n:k}$ is $\begin{bmatrix} V \\ k \end{bmatrix}$, $\dim V = n$. Two vertices (k-subspaces) of $qK_{n:k}$ are adjacent if they are disjoint.

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Theorem (q-analogue of Lovász' theorem)

If $n \geq 2k+1$ and $q \geq 3$, then the chromatic number $\chi(qK_{n:k})$ equals $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. Moreover, each colour class of a minimum colouring is a point-pencil and the points are (one-spaces) in an $(n-k+1)$-dimensional subspace.

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$k = 2$: Ameerah Chowdhury, Chris Godsil, Gordon Royle
$q > q_k$: Tim Mussche

Bose-Burton plus a useful extension

Theorem (Bose-Burton, 66)

If $dim(V) = n$ and $\mathcal{E}$ is a family of 1-spaces in V such that any k-space in V contains an element of $\mathcal{E}$, then $|\mathcal{E}| \geq \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. Equality iff $\mathcal{E} = \begin{bmatrix} H \\ 1 \end{bmatrix}$ for some $(n-k+1)$-space H in V.

In maybe more familiar terms: $\mathcal{E}$ is a (trivial) blocking set for $(k-1)$-spaces in $\mathrm{PG}(n-1, q)$.

Proposition (Useful extension)

If $dim(V) = n$ and $\mathcal{E}$ is a family of $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} - \varepsilon$ one-spaces in V, then the number of k-spaces in V disjoint from all $E \in \mathcal{E}$ is at least $\varepsilon q^{(k-1)(n-k+1)} / \begin{bmatrix} k \\ 1 \end{bmatrix}$.

Proof of the extension

(Projective formulation: In $\mathrm{PG}(n-1,q)$ we have $\mathcal{E}$, a set of $\left[{n-k+1 \atop 1}\right] - \varepsilon$ points. To show: there are at least $\varepsilon q^{(k-1)(n-k+1)}/\left[{k \atop 1}\right]$ disjoint $k(-1)$-spaces).

Induction on k. For $k=1$ there is nothing to prove. Next, let $k>1$ and count incident pairs (1-space, k-space), where the k-space is disjoint from all $E \in \mathcal{E}$:

$$N\begin{bmatrix} k \\ 1 \end{bmatrix} \geq \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} + \varepsilon\right)\varepsilon q^{(k-2)(n-k+1)}/\begin{bmatrix} k-1 \\ 1 \end{bmatrix} \geq$$

$$\geq \varepsilon q^{(k-1)(n-k+1)}.$$

Of course the true value is $\varepsilon q^{(k-1)(n-k)}$, a non-trivial result due to Klaus Metsch using as a main ingredient an algebraic lemma from Zsuzsa Weiner and Tamás Szőnyi.

The chromatic number of q-Kneser

Let G(ood) be the set of centres of point-pencils used in a minimal colouring, B(ad) the other colours.
If $|B| = 0$ then from Bose-Burton we know that $|G|$ is at least $\left[{n-k+1 \atop 1}\right]$ and we know everything.

If $|G| = \left[{n-k+1 \atop 1}\right] - \varepsilon$ then by our extension we find at least $\varepsilon q^{(k-1)(n-k+1)} / \left[{k \atop 1}\right]$ uncoloured k-spaces. Since a bad colour can only be used the size of a H-M family time, and

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k < q^{(k-1)(n-k+1)} / \begin{bmatrix} k \\ 1 \end{bmatrix}$$

for $q \geq 3$, we have a contradiction.

The chromatic number of q-Kneser II. $n = 2k$

The most interesting, but alas also the most difficult case, $qK_{2k:k}$.

Largest cocliques: point pencils and their duals (Godsil, Newman).

Second largest? Conjecture: Hilton-Milner families, and their duals.
Chromatic number $\chi(qK_{2k:k})$? Conjecture: $q^k + q^{k-1}$.
($k = 2$ easy, $k = 3$ Eisfeld-Storme-Sziklai)

Colouring with $q^k + q^{k-1}$ colours:

- points of $\Pi_{k+1} \setminus \Pi_k$
- hyperplanes through Π_k that do not contain Π_{k+1}

PROOF for $q > q_k$.

As $q > q_k$, $\begin{bmatrix} n \\ k \end{bmatrix} \approx q^{(n-k)k}$.

Point-pencils: $\approx q^{k(k-1)}$.

Theorem

Let $\mathcal{F}$ be a maximal coclique in $qK_{2k:k}$ with $\tau(\mathcal{F}) > 1$. Then $\mathcal{F}| \leq cq^{k^2-k-1}$, where $c > 1$.

Sketch of the proof (for $k = 3$): $|HM| = q(q^2 + q + 1)^2 + 1$. If $|\mathcal{F}| > (q + 1)(q^2 + q + 1)^2 \implies \exists L$ s.t. there are more than $q + 1$ elements of $\mathcal{F}$ through L; they generate a 5-space $\implies$ there are at most q^4 planes not meeting the greedy line L.
Leaving out the elements of $\mathcal{F}$ not meeting L we get a system $\mathcal{F}^*$ with $\tau = 2$. Now
$|\mathcal{F}^*| \leq (q + 1)(q^2 + q + 1)(q^2 + q) + (q^3 + q^2 + q + 1)$.
So we get by putting together everything that

$$|\mathcal{F}| \leq (q + 1)(q^2 + q + 1)^2 + q^4(-q^3 - q^2 - q),$$

if $\mathcal{F}$ is not contained in a point-pencil or its dual.

The chromatic number of q-Kneser graphs for $n = 2k$, III.

Trivial bound: $\chi \geq \begin{bmatrix} 2k \\ k \end{bmatrix} / \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix} = q^k + 1$.

Proposition

If in the colouring only point and hyperplane colours are used then we need at least $q^k + q^{k-1}$ colours.

Now suppose we have $q^k + q^{k-1} - \varepsilon$ point/hyperplane colours (out of $q^k - q^{k-1}$ or less).

Remark

If the second largest coclique has size at most $cq^{k^2-k-1} \leq \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}/2$, then $\varepsilon \leq 2q^{k-1}$.

A bipartite graph

Define a bipartite graph in the following way:
Upper points N: k-spaces **not** coloured by the $q^k + q^{k-1} - \varepsilon$ point/hyperplane colours.
$|N| \leq \varepsilon c q^{k^2-k-1}$.

Lower points U: k-spaces that are **uniquely** coloured

Edge: intersect in a $(k-1)$-dim. subspace

Lemma

$$|U| \geq 2\left(\begin{bmatrix} 2k \\ k \end{bmatrix} - |N|\right) - (q^k + q^{k-1} - \varepsilon)\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}.$$

Since $\varepsilon \leq 2q^{k-1}$ and $|N| \leq \varepsilon c q^{k^2-k-1}$, we get that $|U| \approx \begin{bmatrix} 2k \\ k \end{bmatrix}$.

Point degrees, final contradiction

Lemma

The degree of a point in U is at least εq^{k-1}.

Lemma

The degree of a point of N is at most $\begin{bmatrix} k \\ 1 \end{bmatrix}\left(\begin{bmatrix} k+1 \\ 1 \end{bmatrix} - 1\right)(\approx q^{2k-1})$.

FINAL CONTRADICTION

$(q^{k^2} - 2q^{k^2-1})\varepsilon q^{k-1} \leq$ no. of edges $\leq \varepsilon c q^{k^2-k-1} q. \implies \varepsilon = 0.$

Sketch of the proofs (about degrees)

1) (degrees in U) Let P denote the point-colours, H the hyperplane-colours. Let $\pi \in U$ coloured by $p_0 \in \pi$. Then

$$(|P|-1)q^{k-1} + |H|q^{k-1} + q^{k-1}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} \geq q^{k-1}\left(\begin{bmatrix} k+1 \\ 1 \end{bmatrix} - 1\right).$$

LHS counts (upper bounds) no. of k-spaces coloured by $P \setminus \{p_0\}$ or H, RHS counts total no. of k-spaces not through p_0 meeting π in a $(k-1)$-space. $\implies$ degree of π is at least εq^{k-1}

2) (degrees in N) Let $\pi \in N$. Choose a $(k-1)$-space inside, and a (not coloured) k-space containing it:
$(\leq)\begin{bmatrix} k \\ k-1 \end{bmatrix} \cdot \begin{bmatrix} 2k-k+1 \\ k-k+1 \end{bmatrix}$.

THANK YOU

THANK YOU FOR YOUR ATTENTION