Consider a positive integer \( n, q \) the power of a prime, \( V \) an \( n \)-dimensional vector space over \( \mathbb{F}_q \), and \( T \) a linear operator on \( V \). If there exists a vector \( v \in V \) such that \( S = \langle v, Tv, \ldots, T^{k-1}v \rangle \), then \( S \) is called a cyclic subspace of \( V \), and \( v \) is a cyclic vector of \( S \). If \( V \) is cyclic, then \( T \) is a cyclic transformation. A subspace \( S \) of \( V \) is \( T \)-invariant iff \( TS \subseteq S \). The \( T \)-invariant subspaces of \( V \) form a lattice \( L(T) \) which was studied by L. Brickman and P. A. Fillmore in their paper The invariant subspace lattice of a linear transformation, Can. J. Math. 19 (1967), 810–822. Here we quote some of their results:

1. If \( T_i \) denotes the restriction of \( T \) to \( V_i \), then \( L(T) \) is the direct sum of the lattices \( L(T_i) \).

2. \( L(T_i) \) is either simple or a chain.

3. \( L(T_i) \) is a chain if and only if \( V_i \) is cyclic.

4. \( L(T) \) is self-dual, i.e. there exists a bijection \( L(T) \to L(T) \) which reverses the partial order.

We want to determine the number of all \( T \)-invariant subspaces of \( V \). Because of 1. it is enough to study each of the lattices \( L(T_i) \) of the different primary components. Because of 4. it is enough to determine the number of \( k \)-dimensional \( T \)-invariant subspaces of \( V_i \) only for \( 0 \leq k \leq \lfloor (\dim V_i + 1)/2 \rfloor \).

In general the subspaces \( V_i \) are not cyclic themselves, but still must be decomposed into cyclic subspaces. For doing this we were following the ideas of J. P. S. Kung presented in The Cycle Structure of a Linear Transformation over a Finite Field, Linear Algebra and its Applications 36 (1981), 141–155. This decomposition reflects the block diagonal structure of the Jacobi normal form of matrices.

If \( f_i \) annihilates \( V_i \), i.e. if \( c_i = 1 \), then G. E. Seguin’s paper The Algebraic Structure of Codes Invariant under a Permutation, Lecture Notes in Computer Science 1133 (1996), 1–18, describes how to determine the number of invariant subspaces by generalizing the well known formula \( \sum_{k=0}^{n} \left( \prod_{j=0}^{k-1} q^{n-j} - q^j \right) q^k \) for the number of all \( k \)-dimensional subspaces of \( V \).

This method was now generalized in order to determine the number of invariant subspaces also in situations where the minimal polynomial of \( T_i \) is \( f_i^{c_i} \) and \( c_i > 1 \).

By an application of the Cauchy–Frobenius Lemma the number of (monomial) isometry classes of linear \((n,k)\)-codes over \( \mathbb{F}_q \) is the average number of \( T \)-invariant \( k \)-dimensional subspaces of \( V \) for all \( T \) in the full monomial group of degree \( n \) over \( \mathbb{F}_q^* \). This approach seems to be the natural approach for counting isometry classes of codes. We were able to extend tables of these numbers which were previously computed using other methods.