Abstract

On the number of invariant subspaces

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Consider a positive integer n, q the power of a prime, V an n-dimensional vector space over $\mathbb{F}_q$, and T a linear operator on V. If there exists a vector $v \in V$ such that $S = \langle v, Tv, \ldots, T^{k-1}v\rangle$, then S is called a cyclic subspace of V, and v is a cyclic vector of S. If V is cyclic, then T is a cyclic transformation. A subspace S of V is T-invariant iff $TS \subseteq S$. The T-invariant subspaces of V form a lattice $L(T)$ which was studied by L. Brickman and P. A. Fillmore in their paper *The invariant subspace lattice of a linear transformation*, Can. J. Math. **19** (1967), 810–822. Here we quote some of their results:

> Let $V = \bigoplus_{i\in I} V_i$ be the primary decomposition of V with respect to T, then each V_i is T-invariant. The primary components V_i correspond to the irreducible monic divisors f_i of the minimal polynomial $\prod_{i\in I} f_i^{c_i}$, $c_i \geq 1$, of T.
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> 1. If T_i denotes the restriction of T to V_i, then $L(T)$ is the direct sum of the lattices $L(T_i)$, $i \in I$.
> 2. $L(T_i)$ is either simple or a chain.
> 3. $L(T_i)$ is a chain if and only if V_i is cyclic.
> 4. $L(T)$ is self-dual, i.e. there exists a bijection $L(T) \to L(T)$ which reverses the partial order.

We want to determine the number of all T-invariant subspaces of V. Because of 1. it is enough to study each of the lattices $L(T_i)$ of the different primary components. Because of 4. it is enough to determine the number of k-dimensional T-invariant subspaces of V_i only for $0 \leq k \leq \lfloor(\dim V_i + 1)/2\rfloor$.

In general the subspaces V_i are not cyclic themselves, but still must be decomposed into cyclic subspaces. For doing this we were following the ideas of J. P. S. Kung presented in *The Cycle Structure of a Linear Transformation over a Finite Field*, Linear Algebra and its Applications **36** (1981), 141–155. This decomposition reflects the block diagonal structure of the Jacobi normal form of matrices.

If f_i annihilates V_i, i.e. if $c_i = 1$, then G. E. Seguin's paper *The Algebraic Structure of Codes Invariant under a Permutation*, Lecture Notes in Computer Science **1133** (1996), 1–18, describes how to determine the number of invariant subspaces by generalizing the well known formula $\sum_{k=0}^{n}\left(\prod_{j=0}^{k-1}\frac{q^n-q^j}{q^k-q^j}x^k\right)$ for the number of all k-dimensional subspaces of V.

This method was now generalized in order to determine the number of invariant subspaces also in situations where the minimal polynomial of T_i is $f_i^{c_i}$ and $c_i > 1$.

By an application of the Cauchy–Frobenius Lemma the number of (monomial) isometry classes of linear (n, k)-codes over $\mathbb{F}_q$ is the average number of T-invariant k-dimensional subspaces of V for all T in the full monomial group of degree n over $\mathbb{F}_q^*$. This approach seems to be the natural approach for counting isometry classes of codes. We were able to extend tables of these numbers which were previously computed using other methods.