Network Codes and q-Analogues of Combinatorial Designs

Axel Kohnert

Sascha Kurz

Dublin September 2009

Bayreuth University Germany

axel.kohnert@uni-bayreuth.de

UNIVERSITÄT BAYREUTH

- Combinatorial Designs
- Network Codes
- Construction

Combinatorial Designs

UNIVERSITÄT
BAYREUTH

Combinatorial Designs

- a set of v points

Combinatorial Designs

- a set of v points
- a set of blocks (block = set of points)

UNIVERSITÄT BAYREUTH

Combinatorial Designs

- a set of v points
- a set of blocks (block = set of points)
- $t-(v,k,\lambda)$ Design

Combinatorial Designs

- a set of v points
- a set of blocks (block = set of points)
- $t-(v,k,\lambda)$ Design
  each block is a $k-$set
  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Combinatorial Designs

- a set of v points

  a, b, c, d, e, f, g
- a set of blocks (block = set of points)

- $t-(v, k, \lambda)$ Design
  each block is a $k-$set
  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Combinatorial Designs

- a set of v points

  a, b, c, d, e, f, g

- a set of blocks (block = set of points)

  $abe, adg, acf, bcg, bdf, cde, efg$

- $t-(v,k,\lambda)$ Design

  each block is a $k-$set

  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

- a set of v points

  a, b, c, d, e, f, g

- a set of blocks (block = set of points)

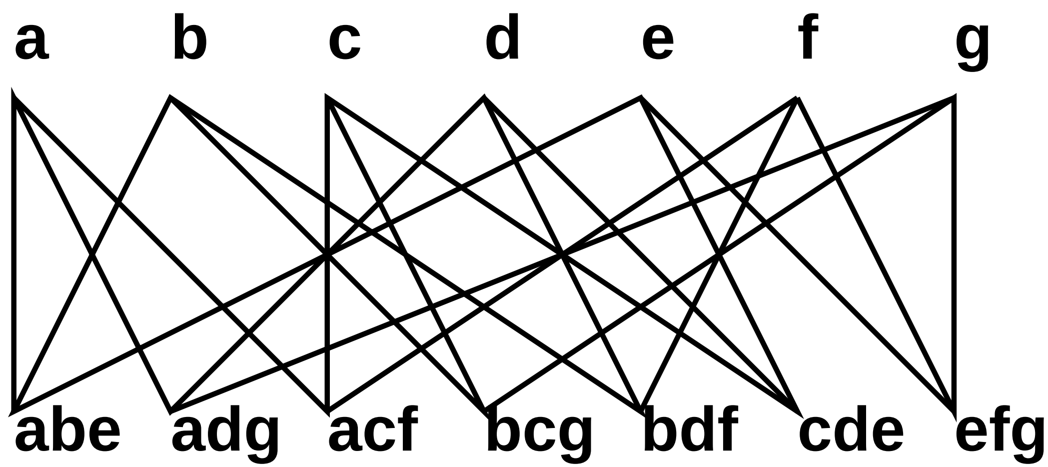
  $abe, adg, acf, bcg, bdf, cde, efg$

- $t-(v,k,\lambda)$ Design
  each block is a $k-$set
  each $t-$set of points is in exactly λ blocks

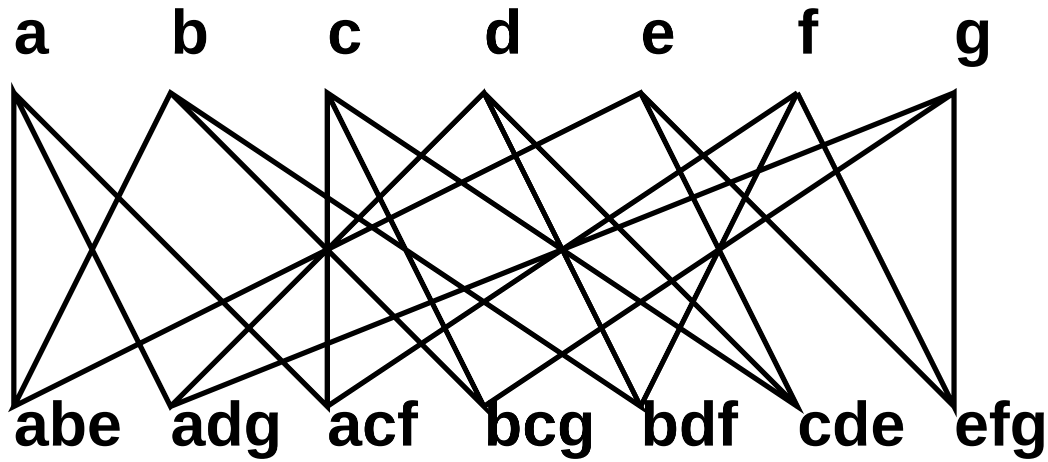
$2-(7,3,1)$ design

UNIVERSITÄT BAYREUTH

Combinatorial Designs



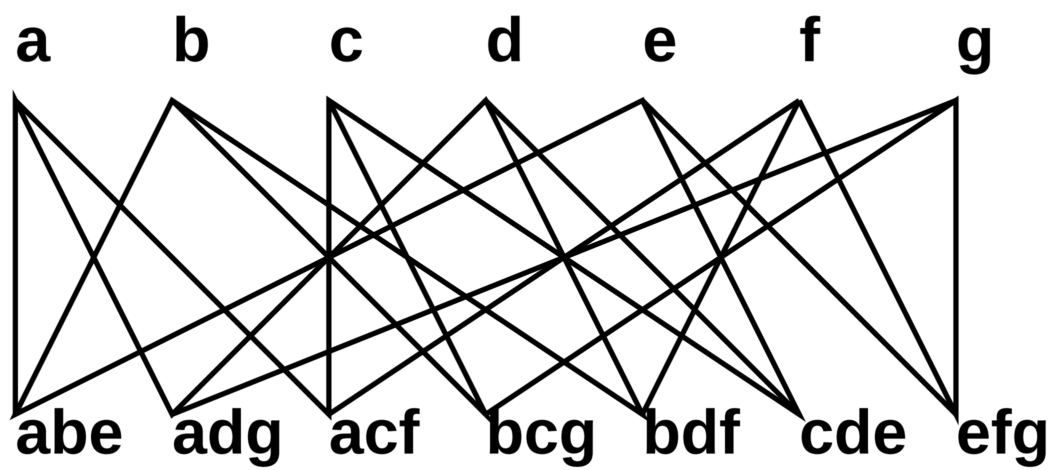
Heawood graph



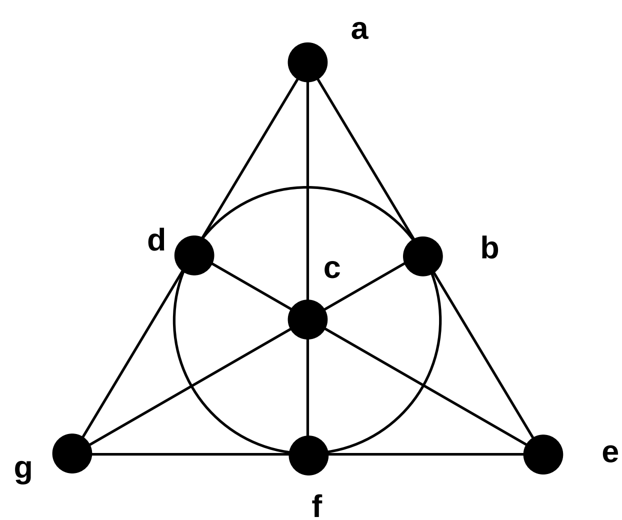
UNIVERSITÄT BAYREUTH

Combinatorial Designs

Heawood graph



Fano plane



UNIVERSITÄT BAYREUTH

Designs over Finite Fields

- a set of v points
- a set of $k-$blocks
- $t-(v,k,\lambda)$ Design
  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs over Finite Fields

- ~~a set of v points~~

  linear $v-$space $GF(q)^v$
- a set of $k-$blocks

- $t-(v,k,\lambda)$ Design

  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs over Finite Fields

- ~~a set of v points~~

  linear $v-$space $GF(q)^v$

- ~~a set of $k-$blocks~~

  a set of $k-$spaces in $GF(q)^v$

- $t-(v,k,\lambda)$ Design

  each $t-$set of points is in exactly λ blocks

UNIVERSITÄT BAYREUTH

Designs over Finite Fields

- ~~a set of v points~~

  linear $v-$space $GF(q)^v$
- ~~a set of $k-$blocks~~

  a set of $k-$spaces in $GF(q)^v$
- $\cancel{t-(v,k,\lambda)}$ ~~Design~~

  ~~each $t-$set of points is in exactly λ blocks~~

  $t-(v,k,\lambda)$ $q-$Design

  each $t-$space of $GF(q)^v$ is in exactly λ of the $k-$spaces

UNIVERSITÄT BAYREUTH

Current State

known:

- Thomas (1987): first to study, $2-$designs
- Braun, Kerber, Laue (2005): first $3-$design

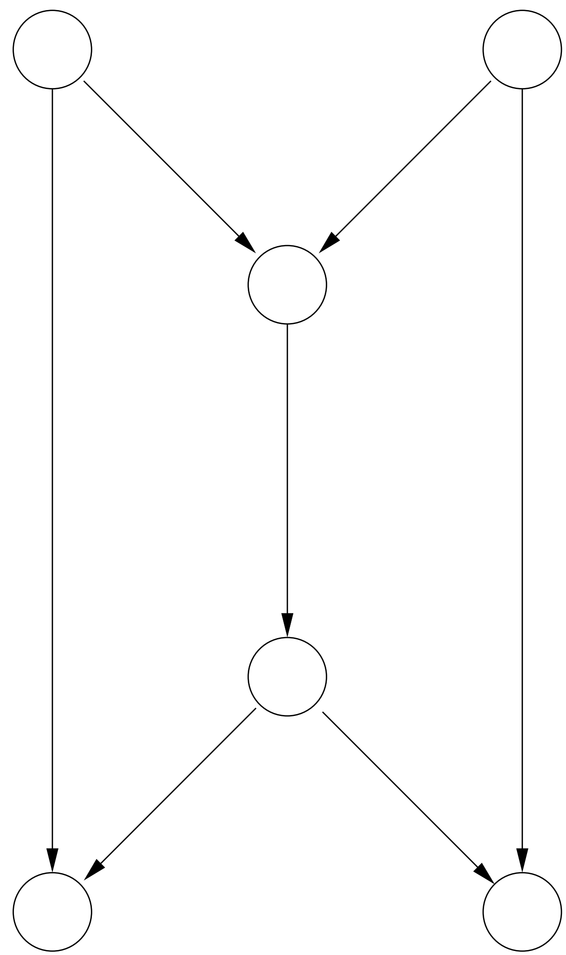
open problems:

- $q-$analog of the Fano plane?
- Steiner systems ? ($\lambda = 1$)
- $t > 3$?

Network Codes

Network Codes

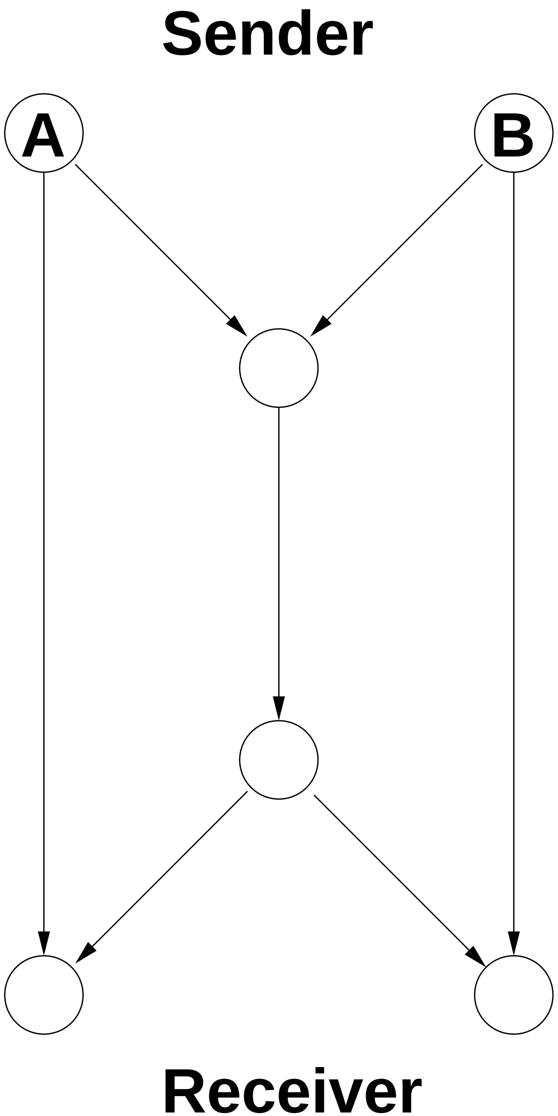
Sender



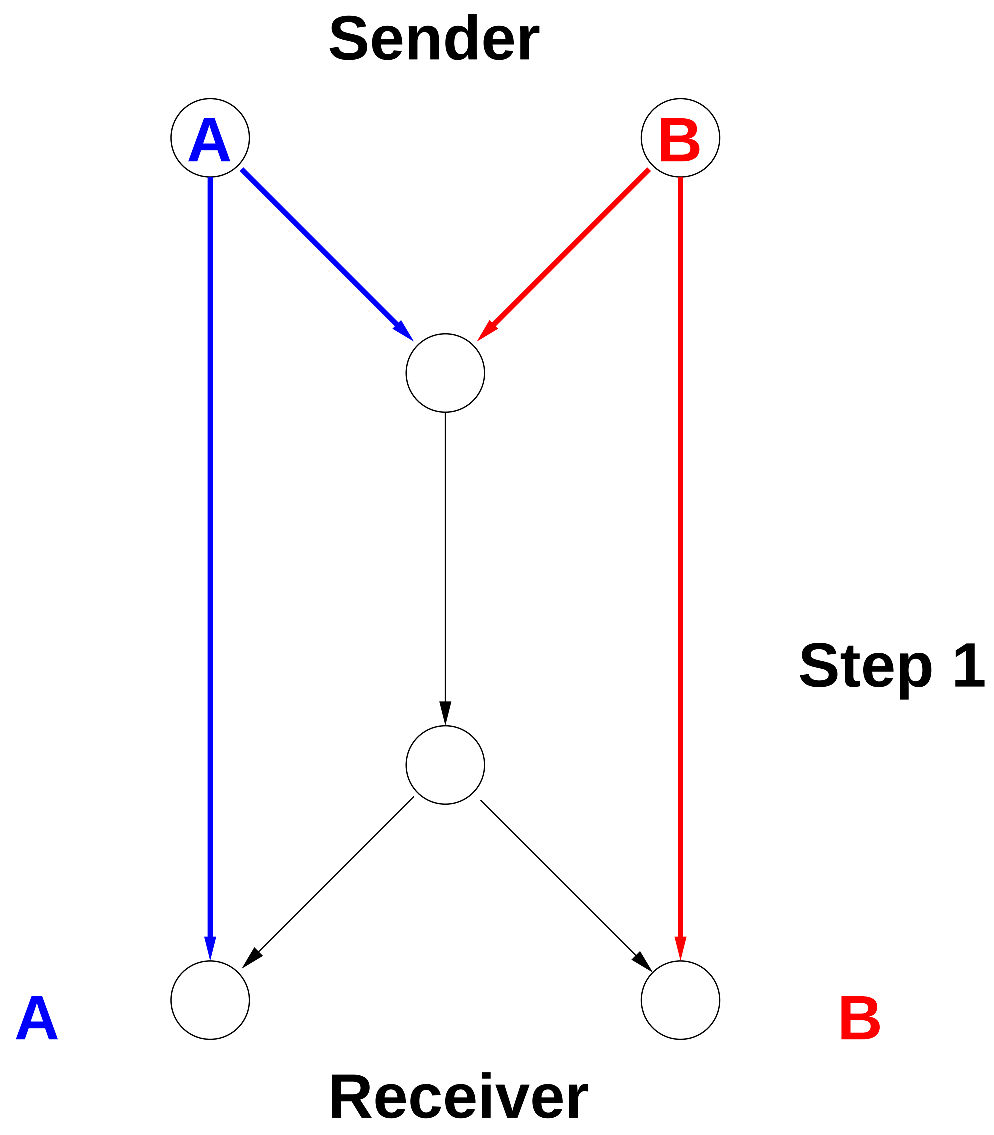
Receiver

UNIVERSITÄT BAYREUTH

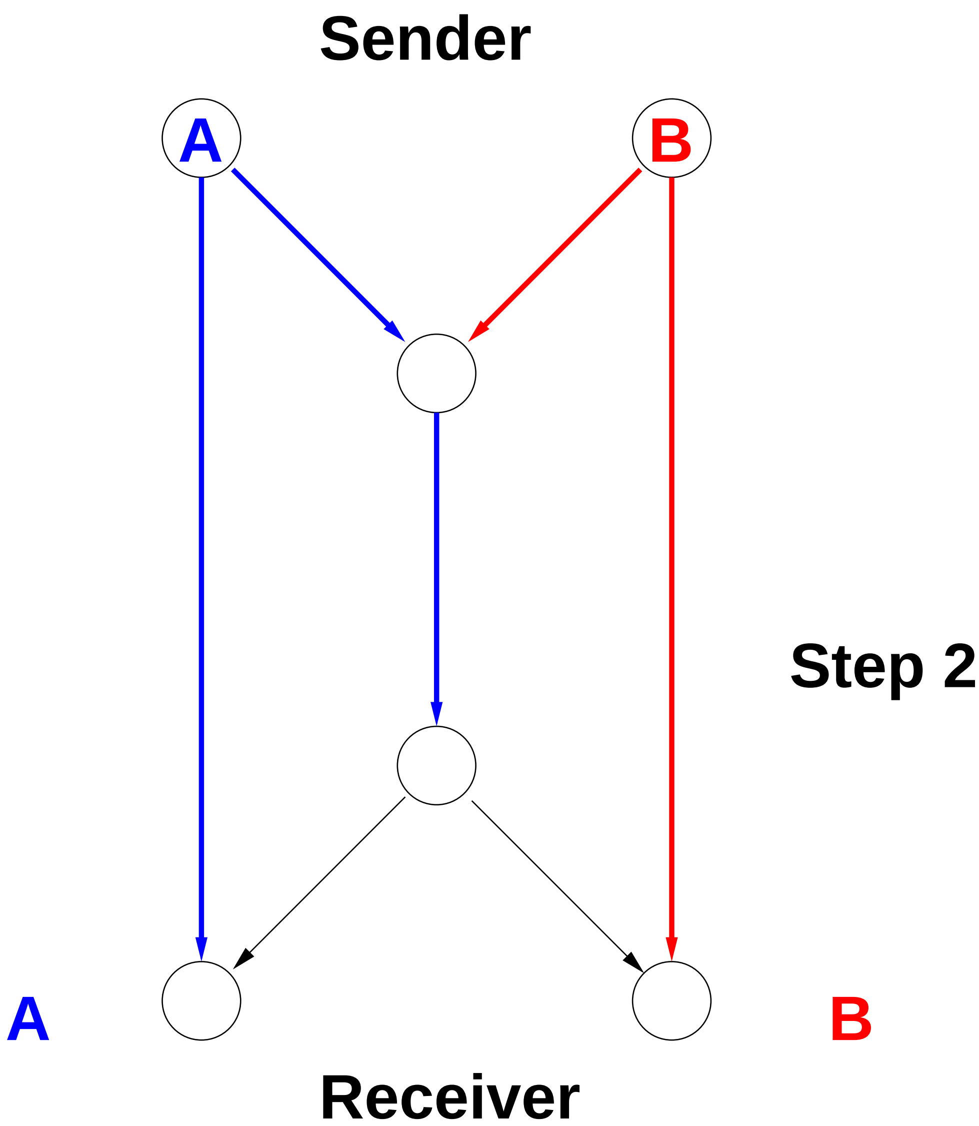
Network Codes



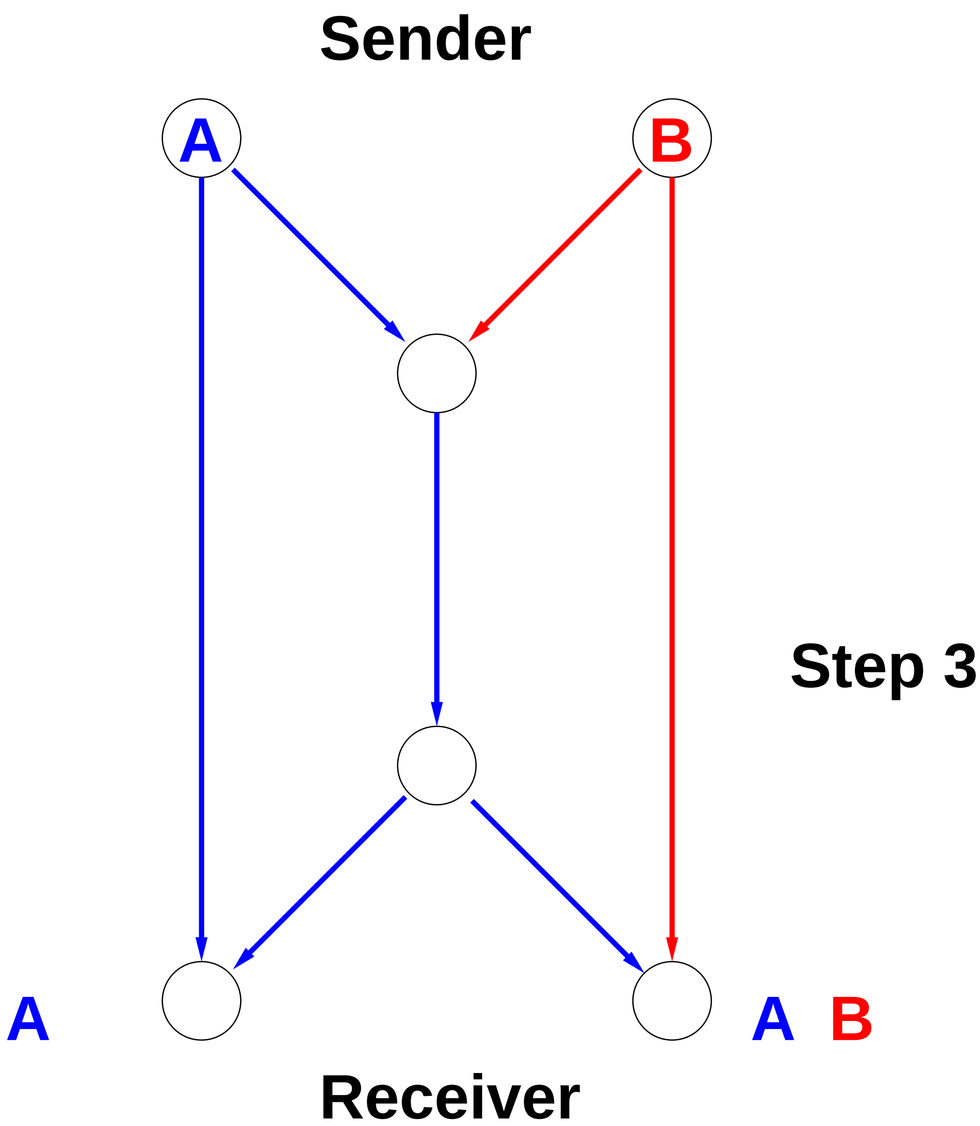
UNIVERSITÄT BAYREUTH


Sender
A
B
Step 1
A
B
Receiver

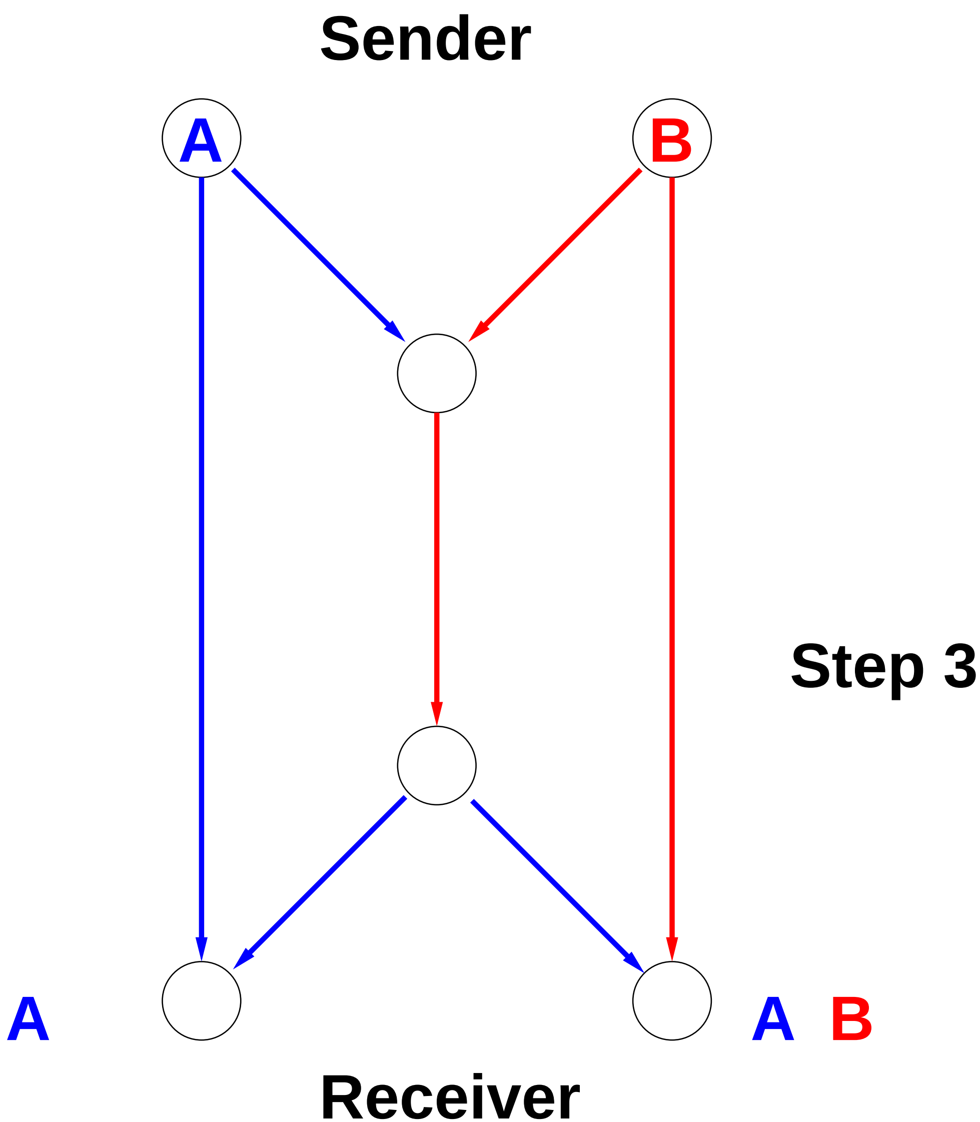
Network Codes



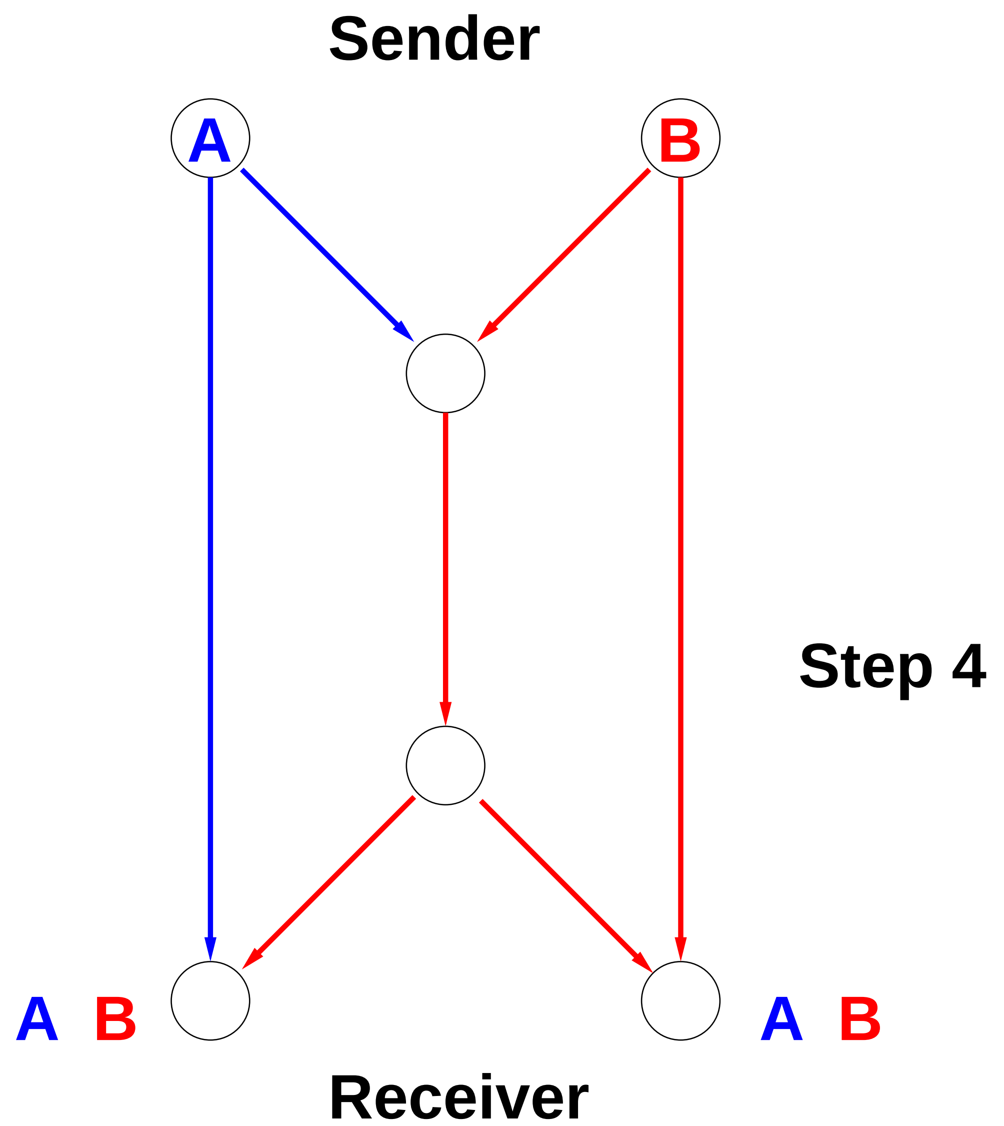
Network Codes



Network Codes

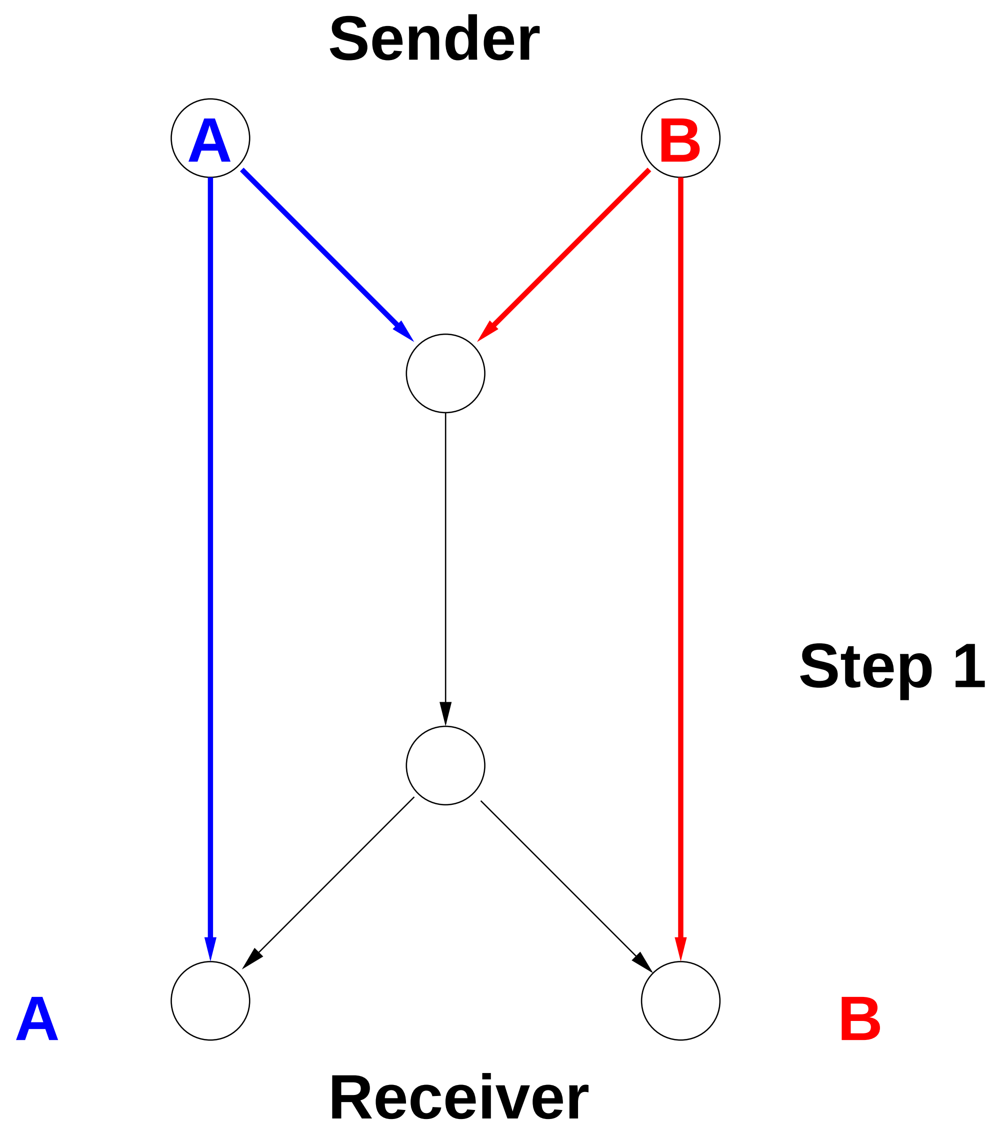


Network Codes

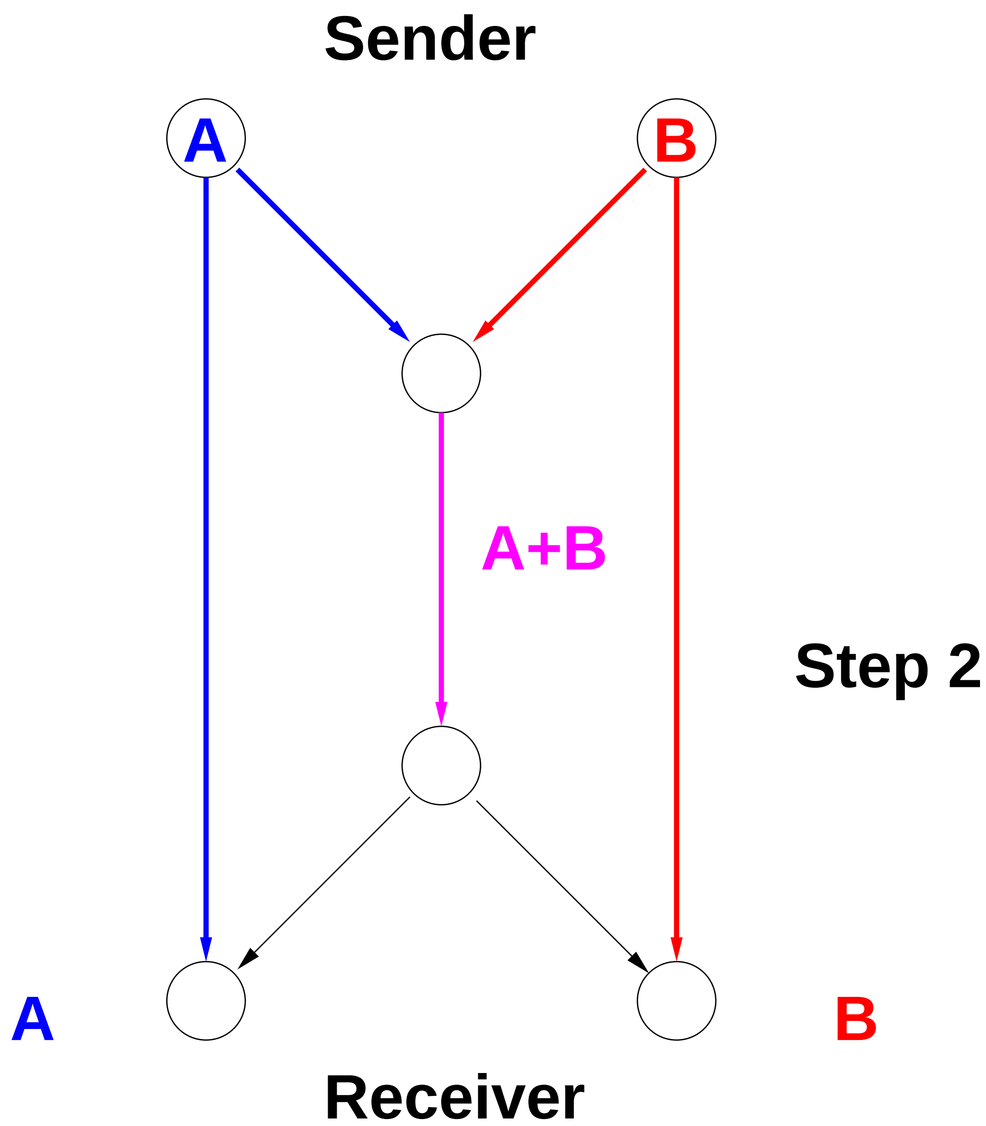


UNIVERSITÄT
BAYREUTH

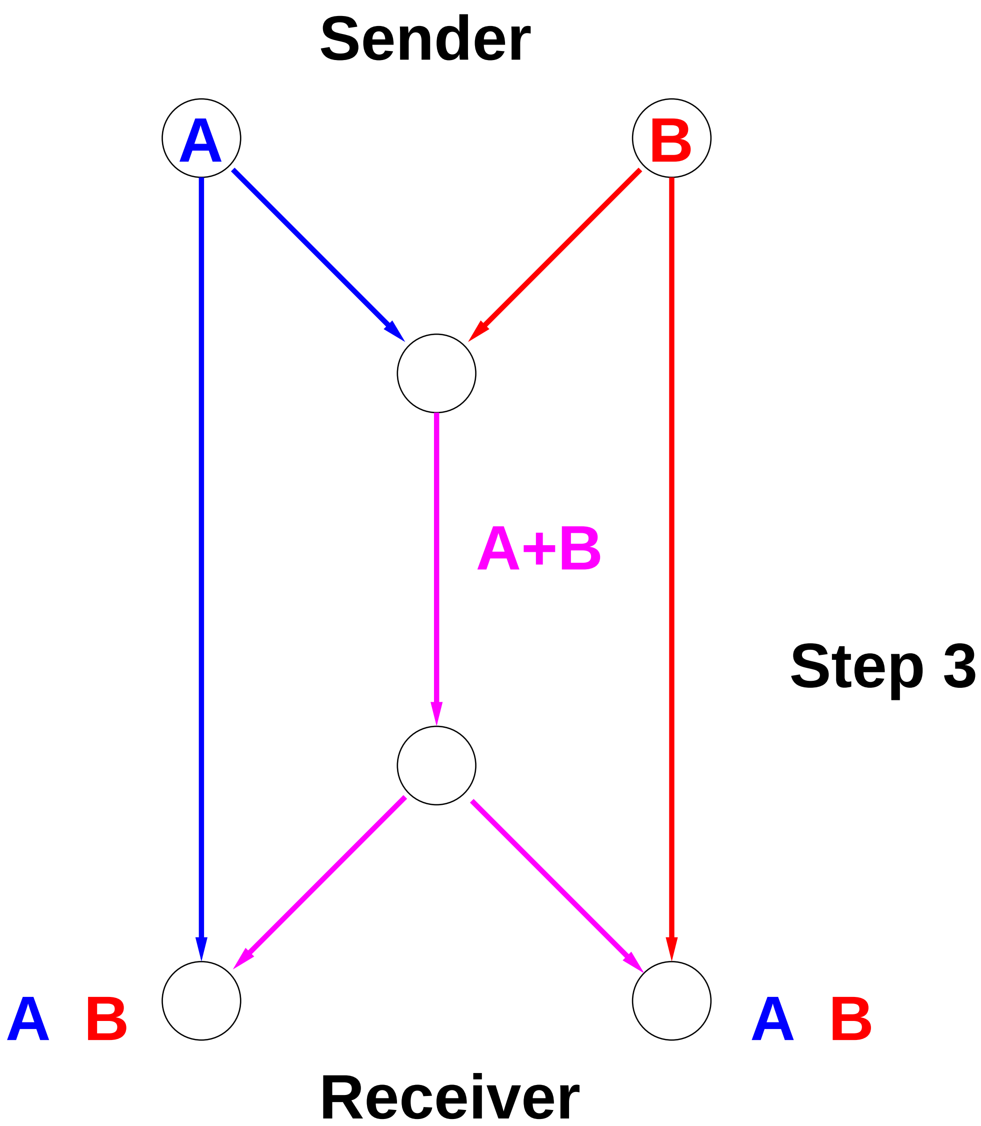
Network Codes



Network Codes



Network Codes



Network Codes

message:

- linear space

Network Codes

message:

- linear space

single node:

- receives vectors
- sends some linear combination of the incoming vectors

UNIVERSITÄT BAYREUTH

Error-Correcting Network Codes

codeword:

- linear subspace of $GF(q)^v$

UNIVERSITÄT BAYREUTH

Error-Correcting Network Codes

codeword:

- linear subspace of $GF(q)^v$

distance d:

- distance in the Hasse diagram of the linear lattice of all subspaces of $GF(q)^v$

Error-Correcting Network Codes

codeword:

- linear subspace of $GF(q)^v$

distance d:

- distance in the Hasse diagram of the linear lattice of all subspaces of $GF(q)^v$

U, W subspace of $GF(q)^v$:

$$d(U,W) = dim(U) + dim(W) - 2dim(U \cap W)$$

UNIVERSITÄT BAYREUTH

Error-Correcting Network Codes

fix minimum distance d:

> Find a set of subspaces in $GF(q)^v$ such that the pairwise distance is at least d

UNIVERSITÄT BAYREUTH

Error-Correcting Network Codes

fix minimum distance d:

Find a set of subspaces in $GF(q)^v$ such that the pairwise distance is at least d

fix also dimension k of the subspaces:

Find a set of $k-$subspaces in $GF(q)^v$ such that the pairwise distance is at least $2d$

UNIVERSITÄT BAYREUTH

Error-Correcting Network Codes

fix minimum distance d:

$$\boxed{\text{Find a set of subspaces in } GF(q)^v \text{ such that the pairwise distance is at least } d}$$

fix also dimension k of the subspaces:

$$\boxed{\text{Find a set of } k-\text{subspaces in } GF(q)^v \text{ such that the pairwise distance is at least } 2d}$$

constant dimension codes $\approx q-$ analogue of constant weight codes

UNIVERSITÄT BAYREUTH

Codes and Designs

Given a $t-(v,k,1)$ $q-$design we get a constant dimension code with minimum distance $2(k-(t-1))$ as the intersection of two codewords has dimension $\leq t-1$.

UNIVERSITÄT BAYREUTH

Codes and Designs

Given a $t-(v,k,1)$ $q-$design we get a constant dimension code with minimum distance $2(k-(t-1))$ as the intersection of two codewords has dimension $\leq t-1$.

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in exactly 1 $k-$subspace
= Steiner system = prefect code

UNIVERSITÄT BAYREUTH

Codes and Designs

Given a $t-(v,k,1)$ $q-$design we get a constant dimension code with minimum distance $2(k-(t-1))$ as the intersection of two codewords has dimension $\leq t-1$.

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in exactly 1 $k-$subspace = Steiner system = prefect code

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in at most 1 $k-$subspace = error-correcting network code

UNIVERSITÄT BAYREUTH

Current State

Define $A_q(v,k,d)$ as the maximal size (= number of codewords) of a constant dimension code with minimum distance d, dimension of codewords = k, and ambient space = $GF(q)^v$

Current State

Define $A_q(v,k,d)$ as the maximal size (= number of codewords) of a constant dimension code with minimum distance d, dimension of codewords = k, and ambient space = $GF(q)^v$

open problems:

- find lower and upper bounds for $A_q(v,k,d)$
- find constructions of ’good’ codes
- special case $A_2(7,3,4)$ = existence of Fano plane

Construction

UNIVERSITÄT
BAYREUTH

Problem

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in at most 1 $k-$subspace
= error-correcting network code

Problem

Find a set of $k-$subspaces in $GF(q)^v$ such that each $t-$subspace is in at most 1 $k-$subspace
= error-correcting network code

D:= incidence matrix between $k-$spaces and $t-$spaces in $GF(q)^v$

$$D_{U,V} := \begin{cases} 1 & t\text{-space } U \text{ is subspace of } k-\text{ space } W \\ 0 & \text{else} \end{cases}$$

UNIVERSITÄT BAYREUTH

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

- $x_1 + \ldots + x_s$ as large as possible

Problem

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

- $x_1 + \ldots + x_s$ as large as possible
- $Dx^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

UNIVERSITÄT BAYREUTH

Combinatorial optimization problem

Find a $0/1$-solution $x = (x_1, \ldots, x_s)$ such that

- $x_1 + \ldots + x_s$ as large as possible
- $Dx^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

solution = network code with minimum distance $2(k - t + 1)$.

UNIVERSITÄT BAYREUTH

Automorphisms

Automorphism φ on $GF(q)^v$: $U < W \iff U^\varphi < W^\varphi$

G subgroup of $Aut(GF(q)^v)$

UNIVERSITÄT BAYREUTH

Automorphisms

Automorphism φ on $GF(q)^v$: $U < W \iff U^\varphi < W^\varphi$
G subgroup of $Aut(GF(q)^v)$

- shrink matrix D by: adding columns of elements in the same orbit of G on the $k-$spaces

$\Rightarrow$ rows of elements in the same orbit on the $t-$spaces are identical

UNIVERSITÄT BAYREUTH

Automorphisms

Automorphism φ on $GF(q)^v$: $U < W \iff U^\varphi < W^\varphi$
G subgroup of $Aut(GF(q)^v)$

- shrink matrix D by: adding columns of elements in the same orbit of G on the $k-$spaces

$\Rightarrow$ rows of elements in the same orbit on the $t-$spaces are identical

- D^G :=shrinked matrix

$\Rightarrow$number of columns = number of orbits on $k-$spaces
number of rows = number of orbits on $t-$spaces

UNIVERSITÄT BAYREUTH

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

UNIVERSITÄT
BAYREUTH

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

- $b_1 x_1 + \ldots + b_m x_m$ as large as possible

UNIVERSITÄT
BAYREUTH

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

- $b_1 x_1 + \ldots + b_m x_m$ as large as possible
- $D^G x^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

UNIVERSITÄT BAYREUTH

Reduced Problem

$b_1, \ldots, b_m$ orbit sizes on $k-$spaces. Find a $0/1$-solution $x = (x_1, \ldots, x_m)$ such that

- $b_1 x_1 + \ldots + b_m x_m$ as large as possible
- $D^G x^T \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

solution = network code with prescribed automorphisms and minimum distance $2(k - t + 1)$.

Results (binary)

v	k	number of codewords: new	old	d
6	3	77	71	4
7	3	304	294	4
8	3	1275	1164	4
9	3	5621	4657	4
10	3	21483	18631	4
11	3	79833	74531	4
12	3	315315	298139	4

Open Problems

- real world $v = 100$
- complete system with encoding and decoding

Last Page

T. Etzion, N. Silberstein: several papers on arxiv.org

A. Kohnert, S. Kurz: *Construction of Large Constant Dimension Codes With a Prescribed Minimum Distance*, LNCS, 2008.

R. Kötter, F. Kschischang: *Coding for errors and erasures in random network coding*, IEEE Transactions on Information Theory, **54**, 3579–3590, 2008.

Thank you very much for for your attention.

UNIVERSITÄT BAYREUTH