Construction of Codes for Network Coding

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(joint work with A.S. Elsenhans, A. Wassermann)

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Agenda

- Network Codes
- Finding Codes (construction)
- Using Codes (decoding)

I - Network Codes

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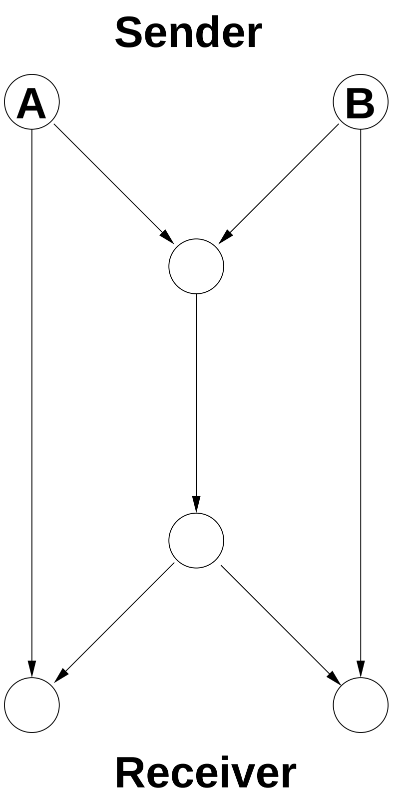
Network Codes

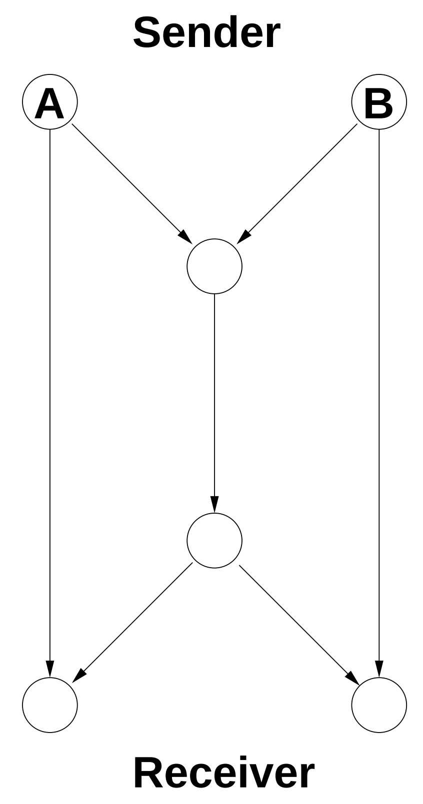
Modell (Kötter, Kschischang)



Network Codes

Modell (Kötter, Kschischang)
one codeword:

- vectorspace $V < \mathbb{F}_2^v$



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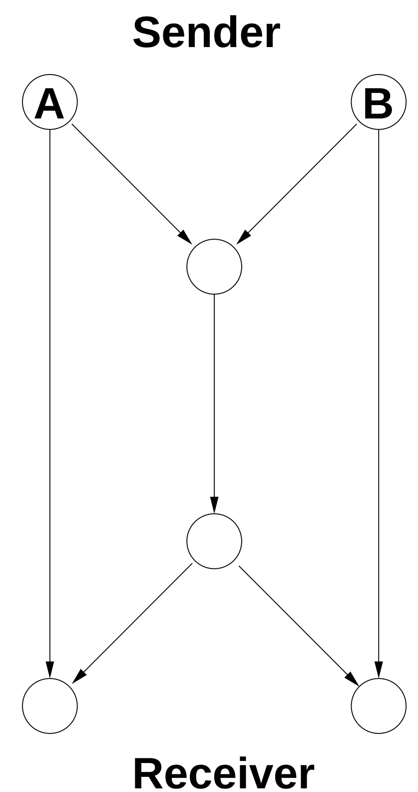
Network Codes

Modell (Kötter, Kschischang)
one codeword:

- vectorspace $V < \mathbb{F}_2^v$

one vertex in the network:

- receives several $v_i \in V$
- sends random combination of the v_i (= EXOR)



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Error Correcting Network Codes

codeword:

- subspace of $\mathbb{F}_2^v$

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$U, W < \mathbb{F}_2^v$:

$$d(U,W) = dim(U) + dim(W) - 2dim(U \cap W)$$

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Error Correcting Network Codes

for a fixed d:

$$\boxed{\text{find a set of subspaces of } \mathbb{F}_2^v \text{ with pairwise distances } \geq d}$$

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constant dimension codes $\approx$ $q-$ analog of constant weight codes

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II - Construction

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Problem

original problem

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modified version

$$\boxed{\text{find } k-\text{dim. subspaces } \{V_1, \dots, V_b\} \text{ in } \mathbb{F}_2^v \text{ such that the pairwise intersection is at most } 1-\text{dimensional}}$$

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$\Rightarrow$ code with minimum distance $\geq 2(k-1)$

Singer Cycle

- On $\mathbb{F}_2^v$ acts the Singer cycle S
- i.e. multiplication in $\mathbb{F}_{2^v}$ with non-zero elements

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Singer Cycle

- On $\mathbb{F}_2^v$ acts the Singer cycle S
- i.e. multiplication in $\mathbb{F}_{2^v}$ with non-zero elements
- inducing action of S on the $k-$spaces

find a Singer orbit O on the $k-$dim. subspaces of $\mathbb{F}_2^v$ such that the pairwise intersection of the $V_i \in O$ is at most $1-$dimensional

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Singer Cycle

- typical Singer orbit on $k-$spaces has $2^v - 1$ elements
- like in the case of the action on $\mathbb{F}_2^v$

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Singer Cycle

- typical Singer orbit on $k-$spaces has $2^v - 1$ elements
- like in the case of the action on $\mathbb{F}_2^v$
- for v large enough there are 'good' orbits having above $1-$dim. intersection property
- good orbit $\Rightarrow$ code with $2^v - 1$ codewords and minimum distance $\geq 2(k-1)$

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Description of Singer orbit

- Given a $k-$dimensional space $V < \mathbb{F}_2^v$
- take all the nonzero vectors $\{u_1, \ldots, u_{2^k-1}\}$
- action of S is multiplication in $\mathbb{F}_{2^v}$

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- action of S is multiplication in $\mathbb{F}_{2^v}$
- pairwise quotients u_i/u_j are invariant under the action of S
- describe a complete orbit by the pairwise quotients

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Example

$k = 3$, $3-$space = $\{0, 1, 4, 10, 18, 23, 25\}$
= exponents of a generator of $\mathbb{F}_{2^v}^*$ (only for the example)

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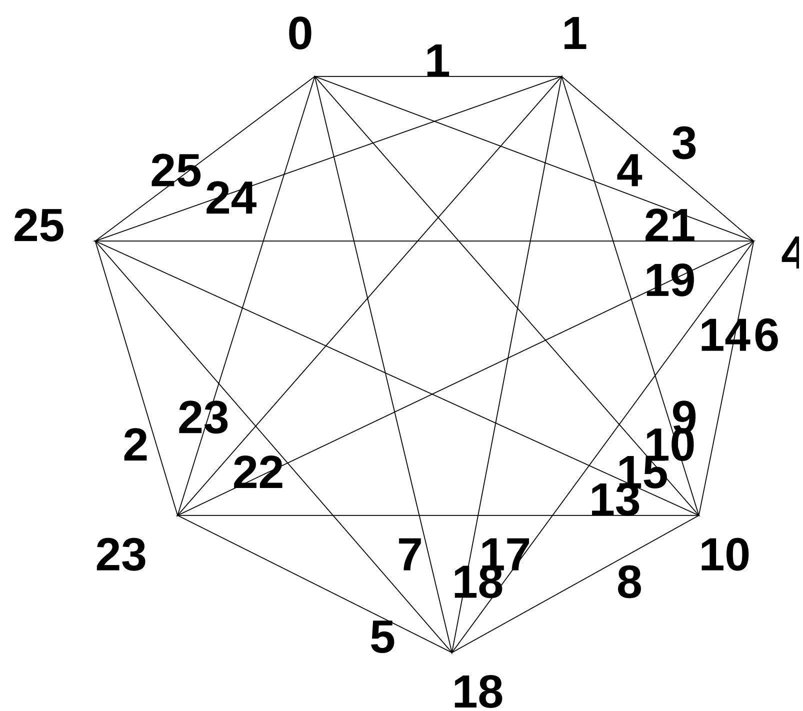
orbit graph G_O



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Lemma: O is a good orbit $\iff$ all the pairwise quotients are different

good orbits

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$$\boxed{\begin{array}{c}\text{find a } k\text{−dim. subspace of } \mathbb{F}_2^v \text{ such that the} \\ \text{pairwise quotients are all different}\end{array}}$$

$\Rightarrow$code with $2^v - 1$ codewords and minimum distance $\geq 2(k-1)$

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good orbits

Lemma: O is a good orbit $\iff$ all the pairwise quotients are different

$$\boxed{\text{find a } k\text{-dim. subspace of } \mathbb{F}_2^v \text{ such that the pairwise quotients are all different}}$$

$\Rightarrow$code with $2^v - 1$ codewords and minimum distance $\geq 2(k-1)$

$$\boxed{\text{find a set } \{V_1, \ldots, V_b\} \text{ of } k\text{-dim. subspace of } \mathbb{F}_2^v \text{ such that all the pairwise quotients are all different}}$$

$\Rightarrow$ code with $b\,(2^v - 1)$ codewords and minimum distance $\geq 2(k-1)$

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results

v	k	b	number of codewords	$d_S = 2d$
15	3	555	$555 \cdot \left(2^{15} - 1\right) = 18185685$	4
16	3	1056	69204960	4
17	3	2108	276297668	4
18	3	4032	1056960576	4

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III - Decoding

Decoding

- special case $b = 1$
- number of codewords $2^v - 1$
- message is a $3-$space $V < \mathbb{F}_2^v$

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Decoding

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as $d = 4$: two possible cases in decoding:

- erasure (we received a $2-$space $U < V$)
- error (i.e. we received a $4-$space $U > V$)

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Erasure

- received a 2$-$space $U = \{x_1, x_2, x_3, 0\} < V$

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- identify an edge $\{x_1, x_2\}$ in orbit graph G_O

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Erasure

- received a $2-$space $U = \{x_1, x_2, x_3, 0\} < V$
- compute x_1/x_2.
- identify an edge $\{x_1, x_2\}$ in orbit graph G_O
- multiply x_1 with an edgelabel u from G_O giving a third base element ux_1 of $V = \langle x_1, x_2, ux_1\rangle$

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Erasure

- received a $2-$space $U = \{x_1, x_2, x_3, 0\} < V$
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- multiply x_1 with an edgelabel u from G_O giving a third base element ux_1 of $V = \langle x_1, x_2, ux_1\rangle$
- costs: one multiplication and one division in $\mathbb{F}_{2^v}$

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Error

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Error

- received a $4-$space $U > V$
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- received a $4-$space $U > V$
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- one of it is a $2-$dim subspace of V and we can apply the erasure algorithm, including a check whether the third constructed vector is in V

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- choose a random $3-$subspace $W < U$, we know: $W \cap V$ is at least $2-$dimensional
- loop over the $7 \quad 2-$dim subspaces of W
- one of it is a $2-$dim subspace of V and we can apply the erasure algorithm, including a check whether the third constructed vector is in V
- worst case costs: 7 divisions and 7 multiplications

Generalisations

- it works for $b > 1$
- it works for $k > 3$

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Thank you

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